Categories and
General Algebraic Structures
with **Applications**

Volume 1, Number 1, December 2013
ISSN Print: 2345-5853 Online: 2345-5861

Shahid Beheshti University
http://www.cgasa.ir

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General Algebraic Structures
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Published by: Shahid Beheshti University

Volume 1, Number 1, December 2013, 27-58
ISSN Print: 2345-5853 Online: 2345-5861

WWW.CGASA.ir

A pointfree version of remainder preservation

Themba Dube and Inderasan Naidoo

Abstract. Recall that a continuous function $f\colon X \to Y$ between Tychonoff spaces is proper if and only if the Stone extension $f^\beta\colon \beta X \to \beta Y$ takes remainder to remainder, in the sense that $f^\beta[\beta X - X] \subseteq \beta Y - Y$. We introduce the notion of "taking remainder to remainder" to frames, and, using it, we define a frame homomorphism $h\colon L \to M$ to be β-proper, λ-proper or υ-proper in case the lifted homomorphism $h^\beta\colon \beta L \to \beta M$, $h^\lambda\colon \lambda L \to \lambda M$ or $h^\upsilon\colon \upsilon L \to \upsilon M$ takes remainder to remainder. These turn out to be weaker forms of properness. Indeed, every proper homomorphism is β-proper, every β-proper homomorphism is λ-proper, and λ-properness is equivalent to υ-properness. A characterization of β-proper maps in terms of pointfree rings of continuous functions is that they are precisely those whose induced ring homomorphisms contract free maximal ideals to free prime ideals.

1 Introduction

Suppose that for each topological space X in some appropriate subcategory of **Top** there is an extension $\varepsilon X \supseteq X$ of X (meaning that a

Keywords: frame, remainder preservation, Stone-Čech compactification, regular Lindelöf coreflection, realcompact coreflection, proper map, lax proper map.
Subject Classification[2000]: 06D22.

homeomorphic copy of X is dense in εX) such that every continuous function $f\colon X \to Y$ between spaces in the subcategory has an extension $f^\varepsilon\colon \varepsilon X \to \varepsilon Y$ which makes the diagram

$$\begin{tikzcd}
\varepsilon X \arrow[r, "f^\varepsilon"] & \varepsilon Y \\
X \arrow[u, "i_X"] \arrow[r, "f"'] & Y \arrow[u, "i_Y"']
\end{tikzcd} \tag{1}$$

commute, where the upward morphisms are embeddings. We say f is ε-*proper* in case f^ε takes remainder to remainder in the sense that $f^\varepsilon[\varepsilon X - X] \subseteq \varepsilon Y - Y$. Throughout, all spaces are assumed to be Tychonoff, that is, completely regular and Hausdorff.

As an example, recall that in Tychonoff spaces proper maps (those continuous functions $f\colon X \to Y$ for which f is closed and the fibers $f^{-1}(y)$ are compact for each $y \in Y$) have several characterizations, including the following:

(a) *For any space Z, the Cartesian product $f \times \mathrm{id}_Z\colon X \times Z \to Y \times Z$ is closed.*

(b) *The square* (1) *above, with ε replaced by β, is a pullback square.*

(c) *f^β takes remainder to remainder, i.e. $f^\beta[\beta X - X] \subseteq \beta Y - Y$.*

The map $f^\beta\colon \beta X \to \beta Y$ in statement (c) is the Stone extension of f. Thus, in the terminology above, β-proper maps in **Tych** are precisely the proper maps – justifying the name "ε-proper".

Our goal is to extend the notion of "taking remainder to remainder" to the category **CRegFrm** of completely regular frames. With β and υ denoting the usual functors (the former assigns the Stone-Čech compactification both in **Tych** and **CRegFrm**, and the latter is the realcompact reflector in **Tych** and the realcompact coreflector in **CRegFrm**), we define β-proper and υ-proper homomorphisms "conservatively" in the sense that, for ε equal to any of these functors, a continuous function

$f\colon X \to Y$ is ε-proper if and only if the induced frame homomorphism $\mathfrak{O}f\colon \mathfrak{O}Y \to \mathfrak{O}X$ is ε-proper. In **CRegFrm** (unlike in **Tych**) there is a Lindelöf coreflector, λ; so we shall also define λ-proper homomorphisms.

Here is a brief overview of the paper. Following this introduction, we recall in Section 2 how the frames βL, λL and υL are constructed. In Section 3 we define β-proper maps, and observe that β-properness is strictly weaker than properness. Recall, from Vermeulen [20], that a frame homomorphism is said to be proper if it is closed and its right adjoint preserves directed joins. Although the definition of β-properness is in terms of the lifted homomorphism $h^{\beta}\colon \beta L \to \beta M$, we have a characterization (Proposition 3.5) in terms of the right adjoint of the map one starts with. This characterization quickly yields that a frame L is compact if and only if the unique homomorphism $\mathbf{2} \to L$ is β-proper (Corollary 3.7). This should be compared with the result in Chen [6] which uses the stronger property of properness to characterize compact frames similarly. Just like proper maps in **Tych** can be characterized in terms of compactifications other than the Stone-Cech compactification (see, for instance, Engelking [11, Theorem 3.7.16]), there is a similar characterization of β-proper maps (Proposition 3.9). We end the section with the ring-theoretic characterization stated in the last sentence in the abstract (Proposition 3.11).

In Section 4 we define λ-proper and υ-proper maps. As in the case of β-properness, these are defined in terms of the lifted homomorphisms $h^{\lambda}\colon \lambda L \to \lambda M$ and $h^{\upsilon}\colon \upsilon L \to \upsilon M$. A pleasant surprise is that the two notions are equivalent (Proposition 4.4), thus enabling us to dispense with the rather recalcitrant functor υ and work mainly with the more accommodating λ in our calculations. Every β-proper map is λ-proper (Corollary 4.3), but not conversely. Indeed, the homomorphism $\mathbf{2} \to L$ is λ-proper if and only if L is realcompact (Proposition 4.5).

The last section casts these notions purely in terms of morphisms, thus giving the discussion a somewhat categorical flavour. The main result (Proposition 5.3) says, for ε equal to β or λ, a homomorphism

$h\colon L \to M$ is ε-proper if and only if the diagram

$$\begin{tikzcd} \varepsilon L \arrow[r, "h^{\varepsilon}"] \arrow[d, "\varepsilon_L"'] & \varepsilon M \arrow[d, "\varepsilon_M"] \\ L \arrow[r, "h"'] & M \end{tikzcd}$$

is a "**2**-pushout" square in the sense that it "pushes out" any wedge of the following form:

$$\begin{tikzcd} & \varepsilon M \arrow[d, "\xi"] \\ L \arrow[r, "\zeta"'] & \mathbf{2} \end{tikzcd}$$

2 Preliminaries

2.1 A brief background on frames

For a general theory of frames we refer to the text by Johnstone [14] and Chapter II in [19] by Picado, Pultr and Tozzi. All frames considered here are assumed to be completely regular. Our notation is standard. For instance, we denote the top element and the bottom element of a frame L by 1_L and 0_L respectively, dropping the subscripts if L is clear from the context. The frame of open subsets of a topological space X is denoted by $\mathfrak{O}X$.

By a *point* of L we mean an element p such that $p \neq 1$ and $x \wedge y \leq p$ implies $x \leq p$ or $y \leq p$. For regular frames, "point" is synonymous with "maximal", where the latter is understood to mean maximal strictly below the top. We denote the set of all points of L by $\mathrm{Pt}(L)$.

We note that

> *if $h\colon L \to M$ is an onto frame homomorphism (between regular frames) and $p \in \mathrm{Pt}(L)$, then either $h(p) = 1$ or $h(p) \in \mathrm{Pt}(M)$.*

Indeed, suppose $h(p) < 1$. Let $y \in M$ be such that $h(p) \leq y < 1$. Then $p \leq h_*(y) < 1$, so that maximality gives $p = h_*(y)$, and hence $h(p) = hh_*(y) = y$. Therefore $h(p) \in \mathrm{Pt}(M)$.

A frame homomorphism is *dense* if it maps only the bottom element to the bottom element, and *codense* if it maps only the top to the top. Any dense homomorphism between regular frames is monic, and any codense homomorphisms between regular frames is one-one. By a *quotient map* we mean an onto frame homomorphism.

An element a of L is a *cozero element* if there is a sequence (a_n) in L such that $a_n \prec\!\!\prec a$ for each n and $a = \bigvee a_n$. The set of all cozero elements of L is called the *cozero part* of L and is denoted by $\mathrm{Coz}\, L$. It is a sub-σ-frame of L which generates L if L is completely regular. For further properties of $\mathrm{Coz}\, L$ and cozero elements, in general, see Banaschewski and Gilmour [3].

2.2 The coreflections βL, λL and υL.

Recall that a full subcategory $\mathbf{C}$ of a category $\mathbf{A}$ is said to be a *coreflective* subcategory if for every object A in $\mathbf{A}$, there is an object γA in $\mathbf{C}$ and a morphism $\gamma_A\colon \gamma A \to A$ such that for any morphism $f\colon C \to A$ with domain in $\mathbf{C}$, there is a unique morphism $\bar{f}\colon C \to \gamma A$ satisfying $\gamma_A \cdot \bar{f} = f$, that is, such that the following triangle commutes:

$$\begin{tikzcd} C \arrow[rr, "\bar{f}"] \arrow[dr, "f"'] & & \gamma A \arrow[dl, "\gamma_A"] \\ & A & \end{tikzcd}$$

The object γA is called the *coreflection* of A.

(a) βL**, the compact, completely regular coreflection.** The category **KCRregFrm** of compact, completely regular frames is a coreflective subcategory of **CRegFrm**, the category of completely regular frames with frame homomorphisms. The compact, completely regular coreflection of L (the frame counterpart of the Stone-Čech compactification of Tychonoff spaces), denoted βL, was first constructed by Banaschewski and Mulvey [5] as the frame of regular ideals of L. It can also be realized as the frame of regular ideals of $\operatorname{Coz} L$ (see, for instance, Banaschewski and Gilmour [4]). For our purposes it is convenient to adopt this latter view. We denote the right adjoint of the join map $\beta_L \colon \beta L \to L$ by r_L, and recall that, in view of the way βL is realized here,

$$r_L(a) = \{c \in \operatorname{Coz} L \mid c \prec\!\prec a\}.$$

(b) λL**, the regular Lindelöf coreflection.** Using localic language, Madden and Vermeer [17] have shown that regular Lindelöf locales form a reflective subcategory of the category of locales by actually constructing the reflection, λL, of any completely regular locale L. We recall the construction in frame terms because that is the category of discourse in this paper.

Let L be a completely regular frame. An ideal of $\operatorname{Coz} L$ is a σ*-ideal* if it is closed under countable joins. The regular Lindelöf coreflection of L, denoted λL, is the frame of σ-ideals of $\operatorname{Coz} L$. The join map $\lambda_L \colon \lambda L \to L$ is a dense quotient map, and is the attendant coreflection map. In fact, this is a special case of a more general result concerning κ-frames (see Madden [16, Proposition 4.4]). We denote by k_L the dense quotient map $k_L \colon \beta L \to \lambda L$ defined by $k_L(I) = \langle I \rangle_\sigma$, where $\langle \cdot \rangle_\sigma$ signifies σ-ideal generation in $\operatorname{Coz} L$.

(c) υL**, the realcompact coreflection**. Recall that a frame L is said to be *realcompact* in case whenever I is a maximal ideal of $\operatorname{Coz} L$ with $\bigvee I = 1$, then $\bigvee S = 1$ for some countable $S \subseteq I$. Realcompact frames are coreflective in **CRegFrm** (see, for instance, Banaschewski and Gilmour [4] and Marcus [18] for details). The realcompact coreflection of L, denoted υL, is constructed in the following manner.

For any $t \in L$, let

$$[t] = \{x \in \operatorname{Coz} L \mid x \leq t\};$$

so that if $c \in \operatorname{Coz} L$, then $[c]$ is the principal ideal of $\operatorname{Coz} L$ generated by c. The map $\ell\colon \lambda L \to \lambda L$ given by

$$\ell(J) = \left[\bigvee J\right] \wedge \bigwedge\{P \in \operatorname{Pt}(\lambda L) \mid J \leq P\}$$

is a nucleus. The frame υL is defined to be $\operatorname{Fix}(\ell)$. We denote by ℓ_L the dense quotient map $\lambda L \to \upsilon L$ effected by ℓ. The join map $\upsilon_L\colon \upsilon L \to L$ is also a dense quotient map. For any L we have

$$\operatorname{Coz}(\lambda L) = \operatorname{Coz}(\upsilon L) = \{[c] \mid c \in \operatorname{Coz} L\},$$

a consequence of which is that each of the maps $\lambda_L\colon \lambda L \to L$ and $\upsilon_L\colon \upsilon L \to L$ is a C-quotient map (see Ball and Walters-Wayland [1] for the definition of a C-quotient map). Also,

$$\operatorname{Pt}(\lambda L) = \operatorname{Pt}(\upsilon L).$$

To see this, recall that if $j\colon M \to M$ is a nucleus, then $\operatorname{Pt}\big(\operatorname{Fix}(j)\big) = \{p \in \operatorname{Pt}(M) \mid j(p) = p\}$. Now let $P \in \operatorname{Pt}(\lambda L)$. Then, for the nucleus $\ell\colon \lambda L \to \lambda L$ defining υL,

$$\ell(P) = \left[\bigvee P\right] \wedge \bigwedge\{Q \in \operatorname{Pt}(\lambda L) \mid P \leq Q\} = \left[\bigvee P\right] \wedge P = P.$$

Therefore $\operatorname{Pt}(\upsilon L) = \operatorname{Pt}(\lambda L)$ since $\upsilon L = \operatorname{Fix}(\ell)$. Lastly, for any $a \in L$,

$$(\lambda_L)_*(a) = (\upsilon_L)_*(a) = [a].$$

3 β-proper maps

We start by motivating how "remainder preservation" can be defined in the most natural way in **CRegFrm** without resorting to categorical machinery. Let $f\colon X \to Y$ be a continuous function between Tychonoff spaces, and let $A \subseteq X$ and $B \subseteq Y$. Define $\mathfrak{a}\colon \mathfrak{O}X \to \mathfrak{O}A$ and $\mathfrak{b}\colon \mathfrak{O}Y \to \mathfrak{O}B$ to be the frame homomorphisms induced by the subspace inclusions $i_A\colon A \to X$ and $i_B\colon B \to Y$ respectively. Recall that

$$(\mathfrak{O}f)_*(U) = Y - \mathrm{cl}_Y\, f[X - U]$$

for each $U \in \mathfrak{O}X$, so that, in view of X being a T_1-space,

$$(\mathfrak{O}f)_*(X - \{x\}) = Y - \{f(x)\}$$

for every $x \in X$.

Lemma 3.1. *Let $f, \mathfrak{a}$ and $\mathfrak{b}$ be as above. Then $f[X - A] \subseteq Y - B$ if and only if $\mathfrak{b}\big((\mathfrak{O}f)_*(p)\big) = 1$ for every $p \in \mathrm{Pt}(\mathfrak{O}X)$ with $\mathfrak{a}(p) = 1$.*

Proof. ($\Rightarrow$) Pick $x \in X$ such that $p = X - \{x\}$. Then $\mathfrak{a}(X - \{x\}) = 1$ implies $A \cap (X - \{x\}) = A$, whence $x \notin A$. So, by the hypothesis, $f(x) \in Y - B$. But this implies $B \cap (Y - \{f(x)\}) = B$, whence $\mathfrak{b}\big((\mathfrak{O}f)_*(p)\big) = 1$.

($\Leftarrow$) Let $z \in X - A$. We must show that $f(z) \in Y - B$. Now, $p = X - \{z\} \in \mathrm{Pt}(\mathfrak{O}X)$ such that $\mathfrak{a}(p) = 1$ since $z \notin A$. So, by the hypothesis, $\mathfrak{b}\big((\mathfrak{O}f)_*(p)\big) = 1$. But, as observed above, $(\mathfrak{O}f)_*(p) = Y - \{f(z)\}$, so $B = B \cap (Y - \{f(z)\})$, which implies $f(z) \notin B$, as required. $\square$

This lemma motivates the following definition. Consider the diagram

$$\begin{tikzcd} L \arrow[r, "h"] \arrow[d, "\mathfrak{a}"'] & M \arrow[d, "\mathfrak{b}"] \\ A & B \end{tikzcd} \tag{2}$$

in **CRegFrm**, where the downward homomorphisms are quotient maps.

Definition 3.2. Let h, $\mathfrak{a}$ and $\mathfrak{b}$ be as in the preceding diagram. We say *h takes $\mathfrak{a}$-remainder to $\mathfrak{b}$-remainder* if $\mathfrak{a}(h_*(p)) = 1$ for every $p \in \mathrm{Pt}(M)$ with $\mathfrak{b}(p) = 1$. If $\mathrm{Pt}(M) = \emptyset$ or $\mathfrak{b}(p) < 1$ for every $p \in \mathrm{Pt}(M)$, we take the requirement of the definition to be vacuously satisfied.

In particular, for any homomorphism $h\colon L \to M$, we shall simply say the Stone extension $h^\beta\colon \beta L \to \beta M$ *takes remainder to remainder* to mean that it takes β_L-remainder to β_M-remainder. Since we are working in **Frm** rather than **Loc**, it would perhaps be more appropriate to talk of a frame homomorphism "co-taking remainder to remainder"; but we do not feel the inclination to be too pedantic about this.

Definition 3.3. We say a frame homomorphism $h\colon L \to M$ is *β-proper* if $h^\beta\colon \beta L \to \beta M$ takes remainder to remainder.

Observe that if L is compact, then there is no point p of βL with $\beta_L(p) = 1$. Thus, any homomorphism into a compact frame is β-proper. It is clear from Lemma 3.1 that, for any continuous function $f\colon X \to Y$, the Stone extension $f^\beta\colon \beta X \to \beta Y$ takes remainder to remainder if and only if $(\mathfrak{O}f)^\beta\colon \beta(\mathfrak{O}Y) \to \beta(\mathfrak{O}X)$ takes remainder to remainder. Consequently, f is a proper map if and only if $\mathfrak{O}f\colon \mathfrak{O}Y \to \mathfrak{O}X$ is β-proper.

Recall that a homomorphism $h\colon L \to M$ is said to be *closed* if $h_*(h(a)\vee b) = a \vee h_*(b)$ for every $a \in L$ and $b \in M$. In [20], Vermeulen defines a homomorphism to be *proper* if it is closed and its right adjoint preserves directed joins. In [15], Korostenski and Labuschagne have called a homomorphism *lax proper* if its right adjoint preserves directed joins. We adhere to this terminology. As we will see, although β-properness is equivalent to properness in the subcategory of **CRegFrm** consisting of the (spatial) frames isomorphic topologies of Tychonoff spaces, it is in general weaker than properness. Indeed, it is implied by lax properness.

For a discussion on proper frame maps the reader should please consult the papers by Chen [6], He and Luo [13] and Vermeulen [20]. In the first of these papers the author adopts the frame version of statement (a) in the characterizations of proper maps of spaces recited in the Introduction as his definition of proper maps of frames (he uses the adjective "perfect"), and establishes its equivalence to the frame version of (b). In none of these papers is the concept of taking remainder to remainder as defined here considered.

Rephrasing, the definition of β-properness says

> $h\colon L \to M$ *is β-proper precisely if* $\bigvee h_*^{\beta}(I) = 1$ *for every* $I \in \mathrm{Pt}(\beta M)$ *with* $\bigvee I = 1$.

Seeing that this involves computation "at the Stone-Čech level", it might be desirable to have a criterion in terms of the right adjoint of the map one starts with. Indeed, such is available. In preparation for showing that, we recall that any homomorphism into a compact frame is proper. The proofs of this fact that we have seen are somewhat roundabout, so we give a direct proof. First note that for regular frames,

> $h\colon L \to M$ *is closed if and only if for every* $a \in L$ *and* $b \in M$, $h(a) \vee b = 1$ *implies* $a \vee h_*(b) = 1$.

Indeed, assume this condition holds and let $x \prec h_*(h(a) \vee b)$. Then $x^* \vee h_*(h(a) \vee b) = 1$, so that, on acting h, we have

$$1 = h(x^*) \vee hh_*(h(a) \vee b)\big) \leq h(x^*) \vee h(a) \vee b = h(x^* \vee a) \vee b.$$

Thus, the stated condition implies $x^* \vee a \vee h_*(b) = 1$, so that $x \leq a \vee h_*(b)$, and hence, by regularity, $h_*(h(a) \vee b) \leq a \vee h_*(b)$, thus implying h is closed as the other inequality holds anyway.

Lemma 3.4. *Any homomorphism into a compact frame is proper.*

Proof. Let $h\colon L \to M$ be a frame homomorphism with M compact. We first show that h is closed. Suppose $h(a) \vee b = 1$ for some $a \in L$ and $b \in M$. By regularity,

$$b \vee \bigvee\{h(x) \mid x \prec a\} = 1,$$

and hence, by compactness of M, $h(t) \vee b = 1$ for some $t \prec a$. Thus $h(t^*) \leq b$, so that $t^* \leq h_*(b)$. But $a \vee t^* = 1$, so $a \vee h_*(b) = 1$, proving closedness. Next, let D be a directed subset of M. Take $z \prec h_*(\bigvee D)$ in L. Then $z^* \vee h_*(\bigvee D) = 1$, which implies $h(z^*) \vee hh_*(\bigvee D) = 1$, and hence $h(z^*) \vee \bigvee D = 1$. By compactness of M, this implies $h(z^*) \vee d = 1$ for some $d \in D$. By closedness, $z^* \vee h_*(d) = 1$, whence $z \leq h_*(d) \leq \bigvee h_*[D]$. By regularity this implies $h_*(\bigvee D) \leq \bigvee h_*[D]$, and hence equality. $\square$

We now give a criterion for β-properness which does not require computation with the right adjoint of the lifted map.

Proposition 3.5. *A homomorphism $h\colon L \to M$ is β-proper if and only if $\bigvee h_*[I] = 1$ for every $I \in \mathrm{Pt}(\beta M)$ with $\bigvee I = 1$.*

Proof. Since the diagram

$$\begin{tikzcd} \beta L \arrow[r, "h^\beta"] \arrow[d, "\beta_L"'] & \beta M \arrow[d, "\beta_M"] \\ L \arrow[r, "h"'] & M \end{tikzcd}$$

commutes, we have $\beta_M \cdot h^\beta = h \cdot \beta_L$, which, on taking right adjoints, yields $h_*^\beta \cdot r_M = r_L \cdot h_*$. Now, for any $I \in \beta M$,

$$I = \bigvee_{\beta M} \{r_M(a) \mid a \in I\},$$

and this join is directed. So, by Lemma 3.4, we have

$$h_*^\beta(I) = \bigvee_{\beta L} \{h_*^\beta r_M(a) \mid a \in I\}.$$

Consequently,

$$\begin{aligned}
\bigvee h_*^\beta(I) &= \beta_L\Big(\bigvee_{\beta L} \{h_*^\beta r_M(a) \mid a \in I\}\Big) \\
&= \beta_L\Big(\bigvee_{\beta L} \{r_L h_*(a) \mid a \in I\}\Big) \\
&= \bigvee_{L} \{h_*(a) \mid a \in I\} \\
&= \bigvee h_*[I].
\end{aligned}$$

The result therefore follows. □

Corollary 3.6. *Lax properness implies β-properness.*

Proof. Let $h\colon L \to M$ be a lax proper homomorphism. Let I be a point of βM with $\bigvee I = 1$. Since $\bigvee I$ is a join of a directed set,

$$\bigvee h_*[I] = h_*\Big(\bigvee I\Big) = h_*(1) = 1,$$

showing that h is β-proper. □

In [6], Chen shows that properness of homomorphisms characterizes compact frames in the sense that a frame L is compact if and only if the unique homomorphism $\pi\colon \mathbf{2} \to L$ is proper. The weaker notion actually suffices.

Corollary 3.7. *A frame L is compact if and only if the homomorphism $\pi\colon \mathbf{2} \to L$ is β-proper.*

Proof. Necessity needs no verification. For the sufficiency, suppose, on the contrary, that L is not compact. Then $\beta_L\colon \beta L \to L$ is not codense, and so there exists $J \neq 1_{\beta L}$ in βL such that $\bigvee J = 1$. Since βL has enough points, there is a point I of βL above J, so that $\bigvee I = 1$. By the hypothesis we have $\bigvee \pi_*[I] = 1$, which implies $\pi_*(u) = 1$ for some $u \in I$, and hence $u = 1$. But this is false since $I < 1_{\beta L}$. □

As one would expect, β-proper maps compose to β-proper maps. Furthermore, if a composite of two β-proper maps is β-proper, then the factor on the left of the composition is β-proper, as we demonstrate in the following corollary.

Corollary 3.8. *Let $h\colon L \to M$ and $g\colon M \to N$ be frame homomorphisms. Then we have:*

1. *If both h and g are β-proper, then $g \cdot h$ is β-proper.*
2. *If $g \cdot h$ is β-proper, then g is β-proper.*

Proof. A routine diagram-chasing using the definition shows that (1) is true. To prove (2), let $I \in \mathrm{Pt}(\beta N)$ with $\bigvee I = 1$. If $g \cdot h$ is β-proper, then $\bigvee (gh)_*[I] = \bigvee h_* g_*[I] = 1$, by Proposition 3.5. Thus,

$$1 = h\Big(\bigvee h_* g_*[I]\Big) = \bigvee h[h_* g_*[I]] \leq \bigvee g_*[I],$$

showing, by Proposition 3.5 again, that g is β-proper. □

It is possible for the composite $g \cdot h$ to be β-proper whilst h is not. For instance, let L be any non-compact completely regular frame with at least one point, say p. Let $\xi \colon L \to \mathbf{2}$ be the homomorphism associated with p. The composite

$$\beta L \xrightarrow{\beta_L} L \xrightarrow{\xi} \mathbf{2}$$

is β-proper (actually, proper) as it maps into a compact frame. The homomorphism $\beta_L \colon \beta L \to L$ is not β-proper. Indeed, since L is not compact, the map β_L is not codense, so there is a point I of βL with $\bigvee I = 1$. But now, since $1 \notin I$,

$$\bigvee (\beta_L)_*[I] = \bigvee \{r_L(u) \mid u \in I\} = \bigcup \{r_L(u) \mid u \in I\} \neq 1_{\beta L}.$$

We now give a characterization of β-proper maps in terms of compactifications. The result extends the spatial characterization of proper maps between Tychonoff spaces as precisely those $f \colon X \to Y$ such that for every compactification αY of Y, the extension $F_\alpha \colon \beta X \to \alpha Y$ of the function f takes remainder to remainder (see, for instance, Engelking [11, Theorem 3.7.16]). An analogous result, from a categorical perspective, is [13, Theorem 1] by He and Luo.

Recall that a *compactification* of L is a dense quotient map $h \colon M \to L$ with M compact regular. We shall frequently write a compactification of L as $\gamma L \to L$, suppressing the name of the homomorphism which we then take to be γ_L, or simply γ, if the subscript is unnecessary. Compactifications of L are compared by saying $\gamma L \to L \ \leq \ \zeta L \to L$ if there is a frame homomorphism $h \colon \gamma L \to \zeta L$ making the triangle

$$\begin{tikzcd} \gamma L \arrow[rr, "h"] \arrow[dr, "\gamma"'] & & \zeta L \arrow[dl, "\zeta"] \\ & L & \end{tikzcd}$$

commute. Being a dense homomorphism out of a regular frame into a compact one, h is then one-one. In particular, for any compactification $\gamma L \to L$ of L there is a homomorphism $\bar{\gamma}\colon \gamma L \to \beta L$ such that $\gamma = \beta_L \cdot \bar{\gamma}$, so that, given any homomorphism $h\colon L \to M$ and any compactification $\gamma L \to L$ of L, the square

$$\begin{tikzcd}
\gamma L \arrow[r, "h^\beta \cdot \bar{\gamma}"] \arrow[d, "\gamma"'] & \beta M \arrow[d, "\beta_M"] \\
L \arrow[r, "h"'] & M
\end{tikzcd}$$

commutes. We now have the following characterization.

Proposition 3.9. *The following are equivalent for a frame homomorphism $h\colon L \to M$.*

1. *h is β-proper.*
2. *For every compactification $\gamma L \to L$ of L, $h^\beta \cdot \bar{\gamma}$ takes γ_L-remainder to β_M-remainder.*
3. *There is a compactification $\zeta L \to L$ of L such that $h^\beta \cdot \bar{\zeta}$ takes ζ-remainder to β_M-remainder.*

Proof. (1) $\Rightarrow$ (2): Let $I \in \mathrm{Pt}(\beta M)$ be such that $\bigvee I = 1$. Then, $\bigvee h_*[I] = 1$, by Proposition 3.5 since h^β takes remainder to remainder. Now the commutativity of the square above implies $h \cdot \gamma = \beta_M \cdot (h^\beta \cdot \bar{\gamma})$, so that $\gamma_* \cdot h_* = (h^\beta \cdot \bar{\gamma})_* \cdot r_M$. Since γ is onto, this implies $h_* = \gamma \cdot (h^\beta \cdot \bar{\gamma})_* \cdot r_M$. Now, for any $u \in I$ we have $r_M(u) \le I$, and hence

$$h_*(u) = \gamma(h^\beta \cdot \bar{\gamma})_* r_M(u) \le \gamma(h^\beta \cdot \bar{\gamma})_*(I).$$

Taking joins over all these u yields

$$1 = \bigvee h_*[I] \leq \gamma(h^\beta \cdot \bar{\gamma})_*(I),$$

which proves that $h^\beta \cdot \bar{\gamma}$ takes γ_L-remainder to β_M-remainder.

(2) $\Rightarrow$ (3): This is trivial.

(3) $\Rightarrow$ (1): Let $p \in \mathrm{Pt}(\beta M)$ with $\beta_M(p) = 1$. Consequently, by the hypothesis, $\zeta\big((h^\beta \cdot \bar{\zeta})_*(p)\big) = 1$. Since $\zeta = \beta_L \cdot \bar{\zeta}$, this implies $\beta_L \bar{\zeta}(\bar{\zeta})_* h^\beta_*(p) = 1$. Hence, $\beta_L\big(h^\beta_*(p)\big) = 1$, since $\bar{\zeta} \cdot (\bar{\zeta})_* \leq \mathrm{id}_{\beta L}$. Therefore h^β takes remainder to remainder. $\square$

We close this section with a ring-theoretic characterization of β-proper maps. We refer to Banaschewski [2] for the definition and properties of the ring $\mathcal{R}L$ of real-valued continuous functions on L. See also the paper by Ball and Walters-Wayland [1] and that by Gutiérrez Garcia, Kubiak and Picado [12]. Here we recall that the coz map links $\mathcal{R}L$ to L in such a way that, for every frame homomorphism $h\colon L \to M$, the induced ring homomorphism $\mathcal{R}h\colon \mathcal{R}L \to \mathcal{R}M$, given by $\mathcal{R}h(\alpha) = h \cdot \alpha$, satisfies $\mathrm{coz}(h \cdot \alpha) = h(\mathrm{coz}\,\alpha)$.

For each $p \in \mathrm{Pt}(\beta L)$, the sets $\mathbf{M}^p$ and $\mathbf{O}^p$ defined by

$$\mathbf{M}^p = \{\alpha \in \mathcal{R}L \mid r_L(\mathrm{coz}\,\alpha) \leq p\} \quad \text{and} \quad \mathbf{O}^p = \{\alpha \in \mathcal{R}L \mid r_L(\mathrm{coz}\,\alpha) \prec\!\!\prec p\}$$

are ideals of $\mathcal{R}L$. We recall, from Dube [8], the following about maximal ideals of $\mathcal{R}L$:

(a) *Maximal ideals of $\mathcal{R}L$ are precisely the ideals $\mathbf{M}^p$ for $p \in \mathrm{Pt}(\beta L)$.*

(b) *Every prime ideal of $\mathcal{R}L$ is contained in a unique maximal ideal. In fact, for every prime ideal of Q of $\mathcal{R}L$ there is a unique $p \in \mathrm{Pt}(\beta L)$ such that $\mathbf{O}^p \subseteq Q \subseteq \mathbf{M}^p$.*

An ideal Q of $\mathcal{R}L$ is said to be *free* if $\bigvee\{\mathrm{coz}\,\alpha \mid \alpha \in Q\} = 1$, in exact analogy with the classical case of $C(X)$. For any ideal H of $\mathcal{R}L$, abbreviate $\{\mathrm{coz}\,\alpha \mid \alpha \in H\}$ by $\mathrm{coz}[H]$. It is shown in [7, Lemma 4.4] that, for any $p \in \mathrm{Pt}(\beta L)$,

$$\bigvee \operatorname{coz}[\mathbf{O}^p] = \bigvee \operatorname{coz}[\mathbf{M}^p] = \beta_L(p).$$

Consequently,

> *a prime ideal of $\mathcal{R}L$ is free if and only if the unique maximal ideal containing it is free.*

In what follows we use subscripts on the **M**-ideals to indicate where the ideal in question resides.

Lemma 3.10. *Let $h\colon L \to M$ be a frame homomorphism, and let $p \in \mathrm{Pt}(\beta M)$. Then $\mathbf{M}_L^{h_*^\beta(p)}$ is the unique maximal ideal of $\mathcal{R}L$ containing the prime ideal $(\mathcal{R}h)^{-1}[\mathbf{M}_M^p]$.*

Proof. Since $\beta_M \cdot h^\beta = h \cdot \beta_L$, and since $\beta_L \cdot r_L = \mathrm{id}_L$, as β_L is onto and r_L is its right adjoint, it follows that $\beta_M h^\beta r_L(a) = h(a)$ for each $a \in L$, and hence

$$h^\beta r_L(a) \leq r_M h(a). \qquad (\dagger)$$

Now let $\alpha \in (\mathcal{R}h)^{-1}[\mathbf{M}_M^p]$. Then $(\mathcal{R}h)(\alpha) = h \cdot \alpha \in \mathbf{M}_M^p$, which implies

$$r_M\big(\operatorname{coz}(h \cdot \alpha)\big) = r_M\big(h(\operatorname{coz}\alpha)\big) \leq p.$$

By (†), this implies $h^\beta r_L(\operatorname{coz}\alpha) \leq p$, so that $r_L(\operatorname{coz}\alpha) \leq h_*^\beta(p)$, and hence $\alpha \in \mathbf{M}_L^{h_*^\beta(p)}$, as required. □

Recall that if $f\colon A \to B$ is a ring homomorphism and I is an ideal of B, then the ideal $f^{-1}[I]$ is called the *contraction* of I by f. Now, in light of the foregoing lemma and the discussion preceding it, it follows easily from the definition that:

Proposition 3.11. *A frame homomorphism is β-proper if and only if the ring homomorphism it induces contracts free maximal ideals to free prime ideals.*

4 λ-proper maps

The functors β, λ and υ share some analogous results with regard to lifted homomorphisms $h^\beta\colon \beta L \to \beta M$, $h^\lambda\colon \lambda L \to \lambda M$ and $h^\upsilon\colon \upsilon L \to \upsilon M$, for a given homomorphism $h\colon L \to M$. For instance, if h satisfies the property that $h^\beta \cdot (\beta_L)_* = (\beta_M)_* \cdot h$, then the result in [9, Proposition 3.8] (see also [10]) states that

$$h \text{ is open} \iff h^\beta \text{ is open} \iff h^\lambda \text{ is open} \iff h^\upsilon \text{ is open.}$$

It is thus natural to investigate the implications that exist between the maps h^β, h^λ and h^υ with regard to remainder preservation. As in the case of the functor β, we introduce the following definitions.

Definition 4.1. A frame homomorphism $h\colon L \to M$ is *λ-proper* if $h^\lambda\colon \lambda L \to \lambda M$ takes λ_L-remainder to λ_M-remainder. It is *υ-proper* if $h^\upsilon\colon \upsilon L \to \upsilon M$ takes υ_L-remainder to υ_M-remainder.

Thus, the definition says

a homomorphism $h\colon L \to M$ is λ-proper if $\lambda_L(h^\lambda_*(p)) = 1$, for any $p \in \mathrm{Pt}(\lambda M)$ with $\lambda_M(p) = 1$.

Observe that if M is Lindelöf, then there is no point p of λM such that $\lambda_M(p) = 1$. Therefore any homomorphism into a Lindelöf frame is automatically λ-proper. Actually there is a stronger statement. Banaschewski and Gilmour in [4, Proposition 1] show that

L is realcompact if and only if $\lambda_L(p) \neq 1$ for each $p \in \mathrm{Pt}(\lambda L)$.

Thus, any homomorphism into a realcompact frame is λ-proper. Consequently, if L is a Lindelöf frame which is not compact, then (in view of Corollary 3.7) the homomorphism $\mathbf{2} \to L$ is a λ-proper map which is not β-proper.

The intent of this section is to investigate the relationships between β-, λ- and υ-properness. As it turns, we shall establish that

$$\text{“}\beta\text{-proper”} \implies \text{“}\lambda\text{-proper”} \iff \text{“}\upsilon\text{-proper”}, \tag{‡}$$

with the first implication not reversible, as already observed. It is because of the equivalence that we have stressed λ-properness in the discussion so far, and even titled the section as we did with no mention of the functor υ in it.

As a first step towards proving the implications in (‡), we establish a quick technical lemma.

Lemma 4.2. *Suppose that in the diagram*

$$\begin{tikzcd}
A \arrow[rrr, "h"] \arrow[dr, "\mathfrak{s}"] \arrow[dd, "\mathfrak{a}"'] & & & B \arrow[dl, "\mathfrak{t}"'] \arrow[dd, "\mathfrak{b}"] \\
& E \arrow[r, "k"] \arrow[dl, "\mathfrak{m}"] & F \arrow[dr, "\mathfrak{n}"] & \\
C & & & D
\end{tikzcd}$$

the downward morphisms are quotient maps, the triangles commute and the trapezoid commutes. If h takes $\mathfrak{a}$-remainder to $\mathfrak{b}$-remainder, then k takes $\mathfrak{m}$-remainder to $\mathfrak{n}$-remainder.

Proof. Let p be a point of F with $\mathfrak{n}(p) = 1$. Then $\mathfrak{t}_*(p) \in \mathrm{Pt}(B)$ and

$$\mathfrak{b}(\mathfrak{t}_*(p)) = \mathfrak{n}\mathfrak{t}(\mathfrak{t}_*(p)) = \mathfrak{n}(p) = 1.$$

So the hypothesis on h implies $\mathfrak{a}(h_*\mathfrak{t}_*(p)) = 1$. We must show that

$\mathfrak{m}(k_*(p)) = 1$. Since

$$1 = \mathfrak{a}h_*\mathfrak{t}_*(p) = \mathfrak{a}\big((\mathfrak{t}h)_*(p)\big) = \mathfrak{a}\big((k\mathfrak{s})_*(p)\big) = \mathfrak{a}\mathfrak{s}_*k_*(p),$$

and since $\mathfrak{m} \cdot \mathfrak{s} = \mathfrak{a}$, it follows that

$$\mathfrak{m}(k_*(p)) = \mathfrak{m}\mathfrak{s}\mathfrak{s}_*k_*(p) = \mathfrak{a}\mathfrak{s}_*k_*(p) = 1.$$

Therefore k takes $\mathfrak{m}$-remainder to $\mathfrak{n}$-remainder. □

Corollary 4.3. *Every β-proper map is λ-proper.*

Proof. In the diagram

$$\begin{tikzcd}
\beta L \arrow[rrr, "h^\beta"] \arrow[dr, "k_L"] \arrow[dd, "\beta_L"'] & & & \beta M \arrow[dl, "k_M"'] \arrow[dd, "\beta_M"] \\
& \lambda L \arrow[r, "h^\lambda"] \arrow[dl, "\lambda_L"] & \lambda M \arrow[dr, "\lambda_M"] & \\
L \arrow[rrr, "h"'] & & & M
\end{tikzcd} \tag{3}$$

the triangles commute, and the upper trapezoid commutes. □

To prove that λ-properness is equivalent to υ-properness, we need to recall from [9, Proposition 3.5] that, for any completely regular frame K, $\lambda_K = \upsilon_K \cdot \ell_K$. Also, as shown in the preliminaries, $\ell_K(p) = p$ for every $p \in \mathrm{Pt}(\lambda K)$, so that $\mathrm{Pt}(\lambda K) = \mathrm{Pt}(\upsilon K)$.

In the following result we also find a condition equivalent to λ-properness which is similar to that in Proposition 3.5 characterizing β-properness.

Proposition 4.4. *For any frame homomorphism $h\colon L \to M$, the following are equivalent.*

1. *h is λ-proper.*

2. *h is υ-proper.*

3. *For any $I \in \mathrm{Pt}(\lambda M)$ with $\bigvee I = 1$, $\bigvee h_*[I] = 1$.*

Proof. (1) $\Rightarrow$ (2): Suppose h is λ-proper. In the diagram

$$\begin{tikzcd}
\lambda L \arrow[rrr, "h^\lambda"] \arrow[dr, "\ell_L"] \arrow[dd, "\lambda_L"'] & & & \lambda M \arrow[dl, "\ell_M"'] \arrow[dd, "\lambda_M"] \\
& \upsilon L \arrow[r, "h^\upsilon"] \arrow[dl, "\upsilon_L"] & \upsilon M \arrow[dr, "\upsilon_M"] & \\
L \arrow[rrr, "h"'] & & & M
\end{tikzcd}$$

each of the triangles commutes, and the lower trapezoid also commutes. It follows therefore by [9, Lemma 3.3] that the upper trapezoid commutes. Thus, by Lemma 4.2, h^υ takes υ_L-remainder to υ_M-remainder; that is, h is υ-proper.

(2) $\Rightarrow$ (1): Suppose h is υ-proper. Let $p \in \mathrm{Pt}(\lambda M)$ be such that $\lambda_M(p) = 1$. We must show that $\lambda_L(h^\lambda_*(p)) = 1$. Since $\lambda_M = \upsilon_M \cdot \ell_M$ and $p = \ell_M(p)$, we have

$$\upsilon_M(p) = \upsilon_M \ell_M(p) = \lambda_M(p) = 1,$$

and hence $\upsilon_L(h^\upsilon_*(p)) = 1$ because h is υ-proper, by the present hypothesis. We claim that $h^\lambda_*(p) = h^\upsilon_*(p)$. Note that the equality $\ell_M(p) = p$ implies $p \le (\ell_M)_*(p)$, and hence $p = (\ell_M)_*(p)$ since points of any regular frame are precisely the maximal elements. Further, the equality

$\ell_M \cdot h^\lambda = h^\upsilon \cdot \ell_L$ implies $h_*^\lambda \cdot (\ell_M)_* = (\ell_L)_* \cdot h_*^\upsilon$. Consequently,

$$h_*^\lambda(p) = h_*^\lambda(\ell_M)_*(p) = (\ell_L)_* h_*^\upsilon(p),$$

which, on applying the onto map ℓ_L, yields

$$h_*^\lambda(p) = \ell_L(h_*^\lambda(p)) = \ell_L(\ell_L)_* h_*^\upsilon(p) = h_*^\upsilon(p).$$

Thus,

$$\lambda_L(h_*^\lambda(p)) = \upsilon_L \ell_L(h_*^\lambda(p)) = \upsilon_L(h_*^\lambda(p)) = \upsilon_L(h_*^\upsilon(p)) = 1,$$

which shows that h is λ-proper.

(1) $\Leftrightarrow$ (3): It suffices to show that $\bigvee h_*^\lambda(I) = \bigvee h_*[I]$ for each $I \in \mathrm{Pt}(\lambda M)$. We show that, in fact, this holds for every $I \in \lambda M$. So let $I \in \lambda M$. Observe that

$$\bigvee_{a \in I} [h_*(a)] = \bigcup_{a \in I} [h_*(a)]$$

because the set on the right is an ideal of $\operatorname{Coz} L$ (as the union is directed), and, in fact, a σ-ideal since for any sequence (s_n) in the set, there is a sequence (t_n) in I such that $s_n \leq h_*(t_n)$ for each n, so that $h\big(\bigvee s_n\big) = \bigvee h(s_n) \leq \bigvee t_n \in I$, and hence $\bigvee s_n \leq h_*\big(\bigvee t_n\big)$, that is, $\bigvee s_n \in [h_*(b)]$ for $b = \bigvee t_n \in I$. We claim that

$$h_*^\lambda(I) = \bigvee_{a \in I} [h_*(a)].$$

Since $\lambda_M \cdot h^\lambda = h \cdot \lambda_L$, we have $h^\lambda_* \cdot (\lambda_M)_* = (\lambda_L)_* \cdot h_*$, so that $h^\lambda_*([a]) = [h_*(a)]$, for any $a \in M$. Now

$$\begin{aligned} h^\lambda\Big(\bigvee\{[h_*(a)] \mid a \in I\}\Big) &= \bigvee\{h^\lambda([h_*(a)] \mid a \in I\} \\ &= \bigvee\{h^\lambda h^\lambda_*([a]) \mid a \in I\} \\ &\leq \bigvee\{[a] \mid a \in I\} \\ &\leq I, \end{aligned}$$

which shows that

$$\bigvee_{a\in I}[h_*(a)] \leq h^\lambda_*(I).$$

On the other hand, let J be any element of λL with $h^\lambda(J) \subseteq I$. For any $u \in J$, $h(u) \in h^\lambda(J) \subseteq I$. Since $u \leq h_* h(u)$, it follows that $u \in \bigcup\limits_{a\in I}[h_*(a)]$, and hence $J \subseteq \bigcup\limits_{a\in I}[h_*(a)]$. Thus,

$$h^\lambda_*(I) \leq \bigvee_{a\in I}[h_*(a)],$$

and hence equality. Therefore

$$\begin{aligned} \bigvee h^\lambda_*(I) = \lambda_L\Big(\bigvee\{[h_*(a)] \mid a \in I\}\Big) &= \bigvee\{\lambda_L([h_*(a)]) \mid a \in I\} \\ &= \bigvee\{h_*(a) \mid a \in I\} \\ &= \bigvee h_*[I], \end{aligned}$$

which completes the proof. □

Reasoning as we did in the β-case, one shows that the composite of λ-proper maps is a λ-proper map, and if the composite $g \cdot h$ is a λ-proper map, then g is λ-proper.

Now, seeing that compact frames are precisely the L for which the map $\mathbf{2} \to L$ is β-proper, one might wonder if Lindelöf frames can be characterized similarly. A look at the proof of Corollary 3.7 shows that the spatiality of βL was crucial. Since, in general, λL is not spatial (even when L is spatial), we cannot adapt that proof. In fact, the result is false, but almost true in the following sense.

Proposition 4.5. *The homomorphism $\mathbf{2} \to L$ is λ-proper if and only if L is realcompact.*

Proof. The "only if" part has already been observed. Conversely, suppose $\pi\colon \mathbf{2} \to L$ is λ-proper. If L is not realcompact, there is a point p of λL with $\lambda_L(p) = 1$. So, in view of π being λ-proper, $\pi_*^\lambda(p)$ is a point of $\lambda\mathbf{2}$ mapped to the top by $\lambda_{\mathbf{2}}$, which is impossible since this homomorphism is an isomorphism. □

Remark 4.6. Apropos of realcompact frames, note that if L is realcompact and $h\colon L \to M$ is λ-proper, then M is realcompact. For, if not, then there is a point I of λM with $\bigvee I = 1$. Then, by virtue of h being λ-proper, $h_*^\lambda(I)$ is a point of λL with join equal to the top, which is impossible since L is realcompact. This strengthens the classical result that if $f\colon X \to Y$ is a proper map between Tychonoff spaces, and Y is realcompact, then X is realcompact. In fact, the classical result is a corollary of the pointfree one because if f is as stated, then $\mathfrak{O}f\colon \mathfrak{O}Y \to \mathfrak{O}X$ is β-proper, and hence a λ-proper map out of a realcompact frame, making $\mathfrak{O}X$ realcompact, and hence X realcompact.

Recall that a frame is *pseudocompact* precisely when every cover by countably many cozero elements has a finite subcover. In particular,

L is pseudocompact if and only if $k_L\colon \beta L \to \lambda L$ is an isomorphism.

Now, a pseudocompact frame does not distinguish between β-properness and λ-properness when it comes to homomorphisms mapping into it. More precisely, we have the following result.

Proposition 4.7. *Any homomorphism into a pseudocompact frame is β-proper if and only if it is λ-proper.*

Proof. The one implication needs no verification. Now suppose that M is a pseudocompact frame and $h\colon L \to M$ is a λ-proper homomorphism. If M is compact, there is nothing to prove. So suppose M is not compact, and let p be a point of βM with $\beta_M(p) = 1$. Then $k_M(p)$ is a point of λM with $\lambda_M\big(k_M(p)\big) = 1$. So, since h is λ-proper, $\lambda_L\big(h_*^\lambda(k_M(p))\big) = 1$. Since in diagram (3) – in the proof of Corollary 4.3 – the triangles commute, and so does the upper trapezoid, we have $k_M \cdot h^\beta = h^\lambda \cdot k_L$, which implies $h_*^\beta \cdot (k_M)_* = (k_L)_* \cdot h_*^\lambda$, and hence $h_*^\beta = (k_L)_* \cdot h_*^\lambda \cdot k_M$, since $(k_M)_* \cdot k_M = \mathrm{id}_{\beta M}$ as k_M is one-one. Consequently, in light of the equality $\lambda_L \cdot k_L = \beta_L$, we have

$$
\begin{aligned}
\beta_L\big(h_*^\beta(p)\big) &= \beta_L(k_L)_* h_*^\lambda(k_M)(p) \\
&= \lambda_L k_L (k_L)_* h_*^\lambda(k_M)(p) \\
&= \lambda_L\big(h_*^\lambda(k_M)(p)\big) \quad \text{since } k_L \text{ is onto} \\
&= 1.
\end{aligned}
$$

Therefore h is β-proper. $\square$

Here is another noteworthy consequence of Lemma 4.2. For brevity, denote by $j_K\colon \beta K \to \upsilon K$ the composite $\ell_K \cdot k_K\colon \beta K \to \lambda K \to \upsilon K$.

Corollary 4.8. *Let $h\colon L \to M$ be a frame homomorphism, and consider the following diagram:*

$$
\begin{tikzcd}
\beta L \arrow[rrr, "h^\beta"] \arrow[dd, "j_L"'] \arrow[dr, "k_L"] & & & \beta M \arrow[dl, "k_M"'] \arrow[dd, "j_M"] \\
& \lambda L \arrow[r, "h^\lambda"] \arrow[dl, "\ell_L"'] & \lambda M \arrow[dr, "\ell_M"] & \\
\upsilon L \arrow[rrr, "h^\upsilon"'] & & & \upsilon M
\end{tikzcd}
$$

The homomorphism h^β takes j_L-remainder to j_M-remainder if and only if it takes k_L-remainder to k_M-remainder.

Proof. Suppose h^β takes j_L-remainder to j_M-remainder. Let $p \in \mathrm{Pt}(\beta M)$ be such that $k_M(p) = 1$. Then $j_M(p) = 1$. So, by the hypothesis, $j_L(h_*^\beta(p)) = 1$. That is, $\ell_L k_L(h_*^\beta(p)) = 1$. Since k_L is onto, either $k_L(h_*^\beta(p) = 1$ or $k_L(h_*^\beta(p) \in \mathrm{Pt}(\lambda L)$. The latter is not possible since ℓ_L maps points to points. Therefore h^β takes k_L-remainder to k_M-remainder.

Conversely, suppose h^β takes k_L-remainder to k_M-remainder. Let p be a point of βM with $j_M(p) = 1$. Since k_M is onto, $k_M(p) = 1$ or $k_M(p) \in \mathrm{Pt}(\lambda M)$. The latter would imply $\ell_M k_M(p) \neq 1$, which is a contradiction. So $k_M(p) = 1$, and hence $k_L(h_*^\beta(p)) = 1$, in light of the current hypothesis. Thus $j_L(h_*^\beta(p)) = 1$, as required. $\square$

5 β- and λ-properness in terms of morphisms

The β- and λ-properness properties, as we have defined them, are couched in a language of both morphisms and elements (or, to be more precise, points). A desirable criterion, from a categorical perspective, should preferably be solely in terms of morphisms. Since there is a one-one correspondence between points of a frame L and homomorphisms $L \to \mathbf{2}$, it is reasonable to expect that such a criterion is available. Our goal in this section is to provide it. To that end, we first note the following lemma.

Lemma 5.1. *Let $h\colon L \to M$ be a quotient map, $p \in \mathrm{Pt}(L)$, and $\xi\colon L \to \mathbf{2}$ the homomorphism associated with p. Then $h(p) \in \mathrm{Pt}(M)$ if and only if there is a homomorphism $\eta\colon M \to \mathbf{2}$ such that the triangle*

$$\begin{tikzcd} L \arrow[rr, "h"] \arrow[dr, "\xi"'] & & M \arrow[dl, "\eta"] \\ & \mathbf{2} & \end{tikzcd}$$

commutes.

Proof. ($\Rightarrow$) Suppose $h(p) \in \mathrm{Pt}(M)$. Let $\eta\colon M \to \mathbf{2}$ be the homomorphism associated with $h(p)$. Now, for any $x \in L$, if $\xi(x) = 0$, then $x \le p$, hence $h(x) \le h(p)$, whence $\eta h(p) = 0$. On the other hand, for any $z \in L$, if $\eta h(z) = 0$, then

$$z \le h_*\eta_*(0) = h_*h(p) = p,$$

whence $\xi(z) = 0$. Therefore $\eta \cdot h = \xi$, so that is η is as required.

($\Leftarrow$) Suppose $\eta\colon M \to \mathbf{2}$ is as postulated. Then $\eta h(p) = \xi(p) = 0$. Therefore $h(p) \neq 1$, and hence $h(p) \in \mathrm{Pt}(M)$ since h is onto. $\square$

Recall diagram (2) preceding Definition 3.2.

Lemma 5.2. *In diagram* (2)*, h takes $\mathfrak{a}$-remainder to $\mathfrak{b}$-remainder if and only if every homomorphism $\xi\colon M \to \mathbf{2}$ factorizes through $\mathfrak{b}\colon M \to B$ whenever the composite map $\xi \cdot h\colon L \to \mathbf{2}$ factorizes through $\mathfrak{a}\colon L \to A$.*

Proof. ($\Rightarrow$) Suppose h takes $\mathfrak{a}$-remainder to $\mathfrak{b}$-remainder, and let $\xi\colon M \to \mathbf{2}$ be a homomorphism such that the composite $\xi \cdot h\colon L \to \mathbf{2}$ factorizes through $\mathfrak{a}\colon L \to A$. So there is a homomorphism $\eta\colon A \to \mathbf{2}$ which makes the triangle

$$\begin{tikzcd} L \arrow[rr, "\mathfrak{a}"] \arrow[dr, "\xi \cdot h"'] & & A \arrow[dl, "\eta"] \\ & \mathbf{2} & \end{tikzcd}$$

commute. Put $p = \xi_*(0)$, so that p is the point of M associated with ξ. Now

$$\eta\mathfrak{a}\big(h_*(p)\big) = \xi h\big(h_*(p)\big) \leq \xi(p) = 0.$$

Therefore $\mathfrak{a}\big(h_*(p)\big) \neq 1$. Since the hypothesis in the implication we are proving is that h takes $\mathfrak{a}$-remainder to $\mathfrak{b}$-remainder, it follows that $\mathfrak{b}(p) \neq 1$. Consequently $\mathfrak{b}(p) \in \mathrm{Pt}(B)$. Since ξ is the homomorphism $M \to \mathbf{2}$ associated with p, it follows from Lemma 5.1 that ξ factorizes through $\mathfrak{b}\colon M \to B$.

($\Leftarrow$) Suppose, by way of contradiction, that h does not take $\mathfrak{a}$-remainder to $\mathfrak{b}$-remainder. Then there is a point p of M with $\mathfrak{b}(p) = 1$, for which $\mathfrak{a}\big(h_*(p)\big) \neq 1$. By Lemma 5.1, if $\xi\colon L \to \mathbf{2}$ is the homomorphism associated with $h_*(p)$, there is a homomorphism $\eta\colon A \to \mathbf{2}$ that makes the triangle

$$\begin{tikzcd} L \arrow[rr, "\mathfrak{a}"] \arrow[dr, "\xi"'] & & A \arrow[dl, "\eta"] \\ & \mathbf{2} & \end{tikzcd}$$

commute. Let $\tau\colon M \to \mathbf{2}$ be the homomorphism associated with p. We claim that $\tau \cdot h = \xi$. To see this, note that, for any $x \in L$, if $\xi(x) = 0$, then $x \leq h_*(p)$, so that $\tau h(x) \leq \tau h h_*(p) \leq \tau(p) = 0$. On the other hand, if z is any element of L with $\tau h(z) = 0$, then $z \leq h_*\tau_*(0) = h_*(p)$, and hence $\xi(z) = 0$. Thus, $\tau \cdot h = \xi$. So τ has the property that $\tau \cdot h\colon L \to \mathbf{2}$ factorizes through $\mathfrak{a}\colon L \to A$. Therefore, by the current hypothesis, there is a homomorphism $\kappa\colon B \to \mathbf{2}$ such that the triangle

$$\begin{tikzcd} M \arrow[rr, "\mathfrak{b}"] \arrow[dr, "\tau"'] & & B \arrow[dl, "\kappa"] \\ & \mathbf{2} & \end{tikzcd}$$

commutes. But this implies $\tau(p) = \kappa\mathfrak{b}(p) = \kappa(1) = 1$, which is false since $\tau(p) = 0$. $\square$

The desired element-free characterization of β-proper and λ-proper maps now follows. Recall how a $\mathbf{2}$-pushout square was described at the end of the Introduction.

Proposition 5.3. *For ε equal to β or λ, any homomorphism $h\colon L \to M$ is ε-proper if and only if the diagram*

$$\begin{tikzcd} \varepsilon L \arrow[r, "h^{\varepsilon}"] \arrow[d, "\varepsilon_L"'] & \varepsilon M \arrow[d, "\varepsilon_M"] \\ L \arrow[r, "h"'] & M \end{tikzcd}$$

is a **2***-pushout square.*

Proof. If the diagram is a **2**-pushout, then h is ε-proper by the last lemma. Conversely, suppose h is ε-proper and consider a diagram of the form

$$\begin{tikzcd} \varepsilon L \arrow[r, "h^{\varepsilon}"] \arrow[d, "\varepsilon_L"'] & \varepsilon M \arrow[d, "\varepsilon_M"'] \arrow[ddr, "\xi"] & \\ L \arrow[r, "h"] \arrow[drr, "\varrho"'] & M \arrow[dr, dashed, "\tau"] & \\ & & \mathbf{2} \end{tikzcd}$$

where the outer quadrilateral commutes, and the dotted line indicates a morphism to be filled in. The composite $\xi \cdot h^{\varepsilon} \colon \varepsilon L \to \mathbf{2}$ factors through $\varepsilon_L \colon \varepsilon L \to L$, so in light of h being an ε-proper map, Lemma 5.2 furnishes a homomorphism $\tau \colon M \to \mathbf{2}$ which makes the upper triangle in the diagram above commute. So it remains to show that the lower triangle also commutes, and that τ is unique with this property. The latter is immediate because if δ also satisfies $\xi = \delta \cdot \varepsilon_M$, then, in light of

$\varepsilon_M \cdot (\varepsilon_M)_* = \mathrm{id}_M$, as ε_M is onto, we have

$$\tau = \xi \cdot (\varepsilon_M)_* = \delta.$$

Let $a \in L$. Then $(\varepsilon_L)_*(a)$ is an element of εL such that, by commutativity of the outer quadrilateral,

$$\xi h^\varepsilon\big((\varepsilon_L)_*(a)\big) = \varrho\varepsilon_L\big((\varepsilon_L)_*(a)\big) = \varrho(a),$$

since $\varepsilon_L \cdot (\varepsilon_L)_* = \mathrm{id}_L$ as ε_L is onto. On the other hand, by commutativity of the inner square and the upper triangle, we have

$$h(a) = h\varepsilon_L(\varepsilon_L)_*(a) = \varepsilon_M h^\varepsilon\big((\varepsilon_L)_*(a)\big),$$

so that

$$\tau h(a) = \tau\varepsilon_M h^\varepsilon(\varepsilon_L)_*(a) = \xi h^\varepsilon\big((\varepsilon_L)_*(a)\big) = \varrho(a).$$

Therefore $\tau \cdot h = \varrho$, showing the desired commutativity, and thus completing the proof. $\square$

Acknowledgement

The authors thank the anonymous referees for remarks which have helped in the revision of the first version of this paper.

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Themba Dube, *Department of Mathematical Sciences, University of South Africa, P.O. Box 392, 0003 Unisa, South Africa.*

Email: dubeta@unisa.ac.za

Inderasan Naidoo, *Department of Mathematical Sciences, University of South Africa, P.O. Box 392, 0003 Unisa, South Africa.*

Email: naidoi@unisa.ac.za