Categories and
General Algebraic Structures
with Applications
Volume 4, Number 1, February 2016, 75-94

The ring of real-continuous functions on a topoframe

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Abstract. A topoframe, denoted by L_τ, is a pair (L, τ) consisting of a frame L and a subframe τ all of whose elements are complementary elements in L. In this paper, we define and study the notions of a τ-real-continuous function on a frame L and the set of real continuous functions $\mathcal{R}L_\tau$ as an f-ring. We show that $\mathcal{R}L_\tau$ is actually a generalization of the ring $C(X)$ of all real-valued continuous functions on a completely regular Hausdorff space X. In addition, we show that $\mathcal{R}L_\tau$ is isomorphic to a sub-f-ring of $\mathcal{R}\tau$. Let τ be a topoframe on a frame L. The frame map $\alpha \in \mathcal{R}\tau$ is called L-*extendable* real continuous function if and only if for every $r \in \mathbb{R}$, $\bigvee^L_{r\in\mathbb{R}}(\alpha(-,r) \vee \alpha(r,-))' = \top$. Finally, we prove that $\mathcal{R}^L\tau \cong \mathcal{R}L_\tau$ as f-rings, where $\mathcal{R}^L\tau$ is the set all of L-extendable real continuous functions of $\mathcal{R}\tau$.

1 Introduction

Pointfree topology (frame theory) focuses on the open sets rather than the points of a space. The ring of real continuous functions in pointfree topology was first created in 1991 by Ball and Hager (see [2]). A systematic and in-depth study of the ring of real continuous functions in pointfree topology was undertaken by B. Banaschewski in the 1997 (see [4]), and subsequently

Keywords: Frame, topoframe, ring of real continuous functions, archimedean ring, f-ring.
Mathematics Subject Classification [2010]: 06D22, 06F25, 54C30, 18B30.
Received: 27 December 2015, Accepted: 10 February 2016
ISSN Print: 2345-5853 Online: 2345-5861

the algebraic aspects of this f-ring studied by T. Dube [5–8] and others [1, 9–12, 14].

In the next section, we review some basic notions and properties of frames and the ring of real functions on a frame τ, denoted by $\mathcal{R}\tau$. In addition, we will present the notion of real-valued functions and also, a topoframe, what we know as modified pointfree topology. Modified pointfree topology focuses on the power set $\mathcal{P}(X)$ accompanied by the open sets $\mathfrak{O}X$ rather than the points of the space X, and deals with abstractly defined as algebraic structures consisting of "a lattice of a power set with a sublattice of open sets", called topoframe.

In Section 3, the concept of τ-real-continuous functions $\mathcal{R}L_\tau$ will be introduced. Then, we show that $\mathcal{R}L_\tau$ is actually a generalization of $C(X)$, the f-ring of all continuous functions from a space X into the set $\mathbb{R}$ (see Theorem 3.3). For more information about $C(X)$, see [13].

In Section 4, we show that $\mathcal{R}L_\tau$ is isomorphic to a sub-f-ring of $\mathcal{R}\tau$ (see Theorem 4.4). Also, the f-ring $\mathcal{R}L_\tau$ is a semiprime and archimedean ring (see Corollary 4.5).

In Section 5, in Theorem 5.4, it is proved that there exists a Boolean algebra B such that τ is a topoframe on B and $\mathcal{R}L_\tau$ is also isomorphic to a sub-f-ring of $\mathcal{R}B_\tau$.

In the last section, the concept of an L-extendable real continuous function is introduced (see Definition 6.4). Finally, in Theorem 6.9, it is proved that $\mathcal{R}^L\tau \cong \mathcal{R}L_\tau$ as f-rings, where $\mathcal{R}^L\tau$ is the set of all L-extendable real continuous functions of $\mathcal{R}\tau$.

2 Preliminaries

In this section, we first gather together the basic facts about frames and rings of real continuous functions on frames, which will be used in the sequel. For further information see [16, 17] on frames and [1, 4] on rings of real functions on frames.

Recall that a *frame* is a complete lattice L in which the distributive law

$$x \wedge \bigvee S = \bigvee \{x \wedge s : s \in S\}$$

holds for all $x \in L$ and $S \subseteq L$. We denote the top element and the bottom element of L by $\top$ and $\perp$, respectively. The frame of open subsets of a topological space X is denoted by $\mathfrak{O}X$.

A subset S of the poset T is said to be dense (or order dense) in T if each element of T is the sup and the inf of some subsets of S. A completion of the poset P is a pair $(C;\varphi)$ where C is a complete lattice and $\varphi : P \longrightarrow C$ is an embedding of P onto a dense subset of C.

A *frame homomorphism* (or a *frame map*) is a map between frames which preserves finite meets and arbitrary joins, including the top and the bottom elements.

The *pseudocomplement* of an element a of a frame L is the element

$$a^* = \bigvee\{x \in L : x \wedge a = \bot\},$$

and the complement of a, if it exists, is denoted by a'.

Proposition 2.1. [17, p. 332] (First De Morgan law). *In a Heyting algebra,*

$$\bigwedge_i a_i^* = (\bigvee_i a_i)^*$$

whenever the supremum $\bigvee_i a_i$ *exists.*

Let L be a frame (locale) and $a, b \in L$. The relation $\prec$ on L given by

$$b \prec a \Leftrightarrow b^* \vee a = 1.$$

The set $\mathcal{R}(L)$ of all frame homomorphisms from $\mathcal{L}(\mathbb{R})$ to L has been widely studied as an f-ring in [4]. Recall that the frame of reals is the frame $\mathcal{L}(\mathbb{R})$ generated by all ordered pairs (p, q), with $p, q \in Q$, subject to the following relations:

(R1) $(p, q) \wedge (r, s) = (p \vee r, q \wedge s)$.

(R2) $(p, q) \vee (r, s) = (p, s)$, whenever $p \leq r < q \leq s$.

(R3) $(p, q) = \bigvee\{(r, s)| : p < r < s < q\}$.

(R4) $\top = \bigvee\{(p, q) : p, q \in \mathbb{Q}\}$.

For $(p, q) \in \mathbb{Q} \times \mathbb{Q}$, we put

$$\langle p, q\rangle := \{x \in \mathbb{Q} : p < x < q\}.$$

and

$$]\!]p, q[\![:= \{x \in \mathbb{R} : p < x < q\}.$$

Corresponding to every continuous operation $\diamond : \mathbb{Q}^2 \to \mathbb{Q}$ (in particular $\diamond \in \{+, . , \wedge, \vee\}$), we have an operation on $\mathcal{R}(L)$, denoted by the same symbol $\diamond$, given by:

$$(\alpha \diamond \beta)(p, q) = \bigvee \{\alpha(r, s) \wedge \beta(u, w) :< r, s > \diamond < u, w > \subseteq < p, q >\}$$

for every $\alpha, \beta \in \mathcal{R}L$, where $< r, s > \diamond < u, w > \subseteq < p, q >$ means that for each $x \in < r, s >$ and $y \in < u, w >$, we have $x \diamond y \in < p, q >$.

We can see in [15] that a *real-valued function on a frame* L is a frame homomorphism $f : \mathcal{P}(\mathbb{R}) \to L$, where one assumes $(\mathcal{P}(\mathbb{R}), \subseteq)$ is a complete Boolean algebra. Let $F(L)$ be the set of all real-valued functions on a frame L. Let $\diamond : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be an operation on $\mathbb{R}$ (in particular, $\diamond \in \{+, . , \wedge, \vee\}$). For every $f, g \in F(L)$, define $f \diamond g : \mathcal{P}(\mathbb{R}) \to L$ by

$$(f \diamond g)(X) = \bigvee \{f(Y) \wedge g(Z) : Y \diamond Z \subseteq X\},$$

where $Y \diamond Z = \{y \diamond z : y \in Y, z \in Z\}$.

Lemma 2.2. [15] *Let f, g be two real-valued functions on a frame L. Then, for every $\diamond \in \{+, ., \vee, \wedge\}$, the following statements hold.*

(1) $(f \diamond g)(X) = \bigvee \{f(\{x\}) \wedge g(\{y\}) : x \diamond y \in X\}$, *for any* $X \in \mathcal{P}(\mathbb{R})$.

(2) $(f \diamond g)(U) = \bigvee \{f(\rrbracket r, s\llbracket) \wedge g(\rrbracket u, v\llbracket) : \rrbracket r, s\llbracket \diamond \rrbracket u, v\llbracket \subseteq U\}$, *for any* $U \in \mathcal{O}(\mathbb{R})$.

(3) $f = g$ *if and only if* $f(\{r\}) = g(\{r\})$, *for any* $r \in \mathbb{R}$.

Theorem 2.3. [15] $(F(L), +, . , \vee, \wedge)$ *is an f-ring.*

As we can see in [18], we managed to modify pointfree topology and talked about topoframes and **TFrm**, the category of topoframes, in the general case. Now, we provide a neat definition of topoframes and topoframe maps as a necessary adjunct to this study and do not deal with it in a considerable detail.

Definition 2.4. A *topoframe* is a pair (L, τ), abbreviated L_τ, consisting of a frame $(L; \wedge, \vee, \perp, \top)$ and a subset τ of L satisfying the following conditions:

1. Every element p of τ has a complement p' in L.

2. The subset τ of L is a subframe of L.

The elements of L belonging to τ are called the *open elements* of L. An element in L is said to be *closed* if it is the complement of an open element. The set of all closed elements is denoted by $\tau' := \{p' : p \in \tau\}$.

Definition 2.5. Let τ_i be a topoframe on a frame L_i, for every $i = 1, 2$. A frame homomorphism $f : L_1 \to L_2$ is called a (τ_1, τ_2)-*homomorphism* if $f(\tau_1) \subseteq \tau_2$.

3 A generalization of $C(X)$

At the start of this section, we show that $\mathcal{R}L_\tau$ is actually a sub-f-ring of $F(L)$. Hereafter, the real line $\mathbb{R}$ is always assumed to be endowed with the natural topology $\mathfrak{O}(\mathbb{R})$.

Definition 3.1. Let τ be a topoframe on a frame L. An $(\mathfrak{O}(\mathbb{R}), \tau)$-homomorphism $f : \mathcal{P}(\mathbb{R}) \to L$ is called a τ-*real-continuous function* on L (or a *real continuous function* on L_τ). The set of all real-continuous functions on L_τ is denoted by $\mathcal{R}L_\tau$.

Theorem 3.2. *For every topoframe L_τ, $\mathcal{R}L_\tau$ is a sub-f-ring of $F(L)$.*

Proof. Obviously, $\mathcal{R}L_\tau \subseteq F(L)$. Let $\diamond \in \{+, ., \vee, \wedge\}$ be an operation on the f-ring $F(L)$. If $f, g \in \mathcal{R}L_\tau$, then for each $p, q \in \mathbb{Q}$, by Lemma 2.2,

$$(f \diamond g)(\rrbracket p, q\llbracket) = \bigvee\{f(\rrbracket r, s\llbracket) \wedge g(\rrbracket u, v\llbracket) : \rrbracket r, s\llbracket \diamond \rrbracket u, v\llbracket \subseteq \rrbracket p, q\llbracket\}$$

belongs to τ, because $f(\mathfrak{O}(\mathbb{R})) \subseteq \tau$ and $g(\mathfrak{O}(\mathbb{R})) \subseteq \tau$. Therefore, $f \diamond g \in \mathcal{R}L_\tau$, by Definition 2.5. Thus, $\mathcal{R}L_\tau$ is a sub-f-ring of $F(L)$. □

That $\mathcal{R}L_\tau$ is a generalization of $C(X)$ is considered in the following theorem.

Theorem 3.3. *The assignment $\theta(f) = f^{-1}$ from $C(X)$ to $\mathcal{R}(\mathcal{P}(X)_{\mathcal{O}(X)})$ is an f-ring isomorphism, where $f^{-1}(A) = \{x \in X \mid f(x) \in A\}$ for all $A \in \mathcal{P}(\mathbb{R})$.*

Proof. Obviously, θ is a function. Let $f, g \in C(X)$ and $f^{-1} = g^{-1}$. Then for every $x \in X$,

$$x \in f^{-1}(\{f(x)\}) = g^{-1}(\{f(x)\}),$$

which follows that $f(x) = g(x)$. Hence $f = g$. Therefore, θ is a one-one function.

To show that θ is an onto function, let $g : \mathcal{P}(\mathbb{R}) \longrightarrow \mathcal{P}(X)$ be a real continuous function on $\mathcal{P}(X)$. Define $h : X \longrightarrow \mathbb{R}$ given by

$$h(x) = \lambda \ \text{ iff } \ x \in g(\{\lambda\})$$

for any $x \in \mathbb{R}$. Since $\{g(\{\lambda\}) \,|\, \lambda \in \mathbb{R},\, g(\{\lambda\}) \neq \emptyset\}_{\lambda\in\mathbb{R}}$ is a partition for X, h is a function with domain X. The equality $\theta(h) = h^{-1} = g$ follows immediately from the definition. The function h is a continuous function since $h^{-1} = g$ and g assigns each open set of $\mathbb{R}$ to an open set of X, by definition.

Suppose $f, g \in \mathbb{R}^X$, $r \in \mathbb{R}$ and $\diamond \in \{+, ., \wedge, \vee\}$. We show that θ preserves all $\diamond$'s.

$$(\theta(f) \diamond \theta(g))(\{r\}) = (f^{-1} \diamond g^{-1})(\{r\}) = \bigcup \{f^{-1}(\{a\}) \cap g^{-1}(\{b\}) \mid a \diamond b = r\},$$

by Lemma 2.2. Moreover,

$$\theta(f \diamond g)(\{r\}) = (f \diamond g)^{-1}(\{r\}) = \{x \in X \mid (f \diamond g)(x) = r\}.$$

Let $z \in (\theta(f) \diamond \theta(g))(\{r\})$. Then there exist $a, b \in \mathbb{R}$ with $a \diamond b = r$ such that $z \in f^{-1}(\{a\}) \cap g^{-1}(\{b\})$, and thus

$$(f \diamond g)(z) = f(z) \diamond g(z) = a \diamond b = r,$$

which follows that $z \in \theta(f \diamond g)(\{r\})$. Hence,

$$(\theta(f) \diamond \theta(g))(\{r\}) \subseteq \theta(f \diamond g)(\{r\}).$$

To establish the reverse inclusion, consider $z \in \theta(f \diamond g)(\{r\})$, then

$$f(z) \diamond g(z) = (f \diamond g)(z) = r\,.$$

Since $z \in f^{-1}(\{f(z)\}) \cap g^{-1}(\{g(z)\})$, we conclude that $z \in (\theta(f) \diamond \theta(g))(\{r\})$. Hence,

$$\theta(f \diamond g)(\{r\}) \subseteq (\theta(f) \diamond \theta(g))(\{r\}).$$

Therefore $\theta(f \diamond g) = \theta(f) \diamond \theta(g)$. This completes the proof. $\square$

4 $\mathcal{R}L_\tau$ is a sub-f-ring of $\mathcal{R}\tau$

The main purpose of this section is to show that $\mathcal{R}L_\tau$ is isomorphic to a sub-f-ring of $\mathcal{R}\tau$. Throughout this paper we assume that j is the frame isomorphism

$$\begin{aligned} j : \mathcal{L}(\mathbb{R}) &\longrightarrow \mathfrak{O}(\mathbb{R}) \\ (p,q) &\longmapsto \rrbracket p,q\llbracket, \end{aligned}$$

and $i : \tau \longrightarrow L$ is the inclusion map. Also, for every $f \in \mathcal{R}L_\tau$, the composition $f \circ j$ from $\mathcal{L}\mathbb{R}$ to L (with L replaced by τ) is denoted by r_f. Evidently, $r_f \in \mathcal{R}\tau$.

Remark 4.1. For every $f \in \mathcal{R}L_\tau$ and $A \subseteq \mathbb{R}$, the complement of $f(A)$ is, by definition, $(f(A))'$, abbreviated $f(A)'$. It is immediately evident from the definition of the complement that $f(A)' = f(\mathbb{R} \setminus A)$.

Proposition 4.2. *Let L be a frame and $f : \mathcal{P}(\mathbb{R}) \to L$ be a frame map. Then f also preserves arbitrary meet.*

Proof. Let $\{A_i\} \subseteq \mathcal{P}(\mathbb{R})$. Then

$$\begin{aligned} f(\textstyle\bigcap_i A_i) &= f((\textstyle\bigcup_i A_i')') \\ &= f(\textstyle\bigcup_i A_i')' && \text{by Remark 4.1} \\ &= (\textstyle\bigvee_i f(A_i'))' \\ &= \textstyle\bigwedge_i f(A_i')' && \text{by Lemma 2.1} \\ &= \textstyle\bigwedge_i f(A_i) && \text{by Remark 4.1} \end{aligned}$$

This completes the assertion. □

Lemma 4.3. *Let L_τ be a topoframe, then for every $f, g \in \mathcal{R}L_\tau$, the following properties hold.*

(1) *The following diagram commutes.*

$$\begin{tikzcd}
\mathcal{P}(\mathbb{R}) \arrow[r, "f"] & L \\
\mathcal{L}(\mathbb{R}) \arrow[u, "j"] \arrow[r, "r_f"] & \tau \arrow[u, "i"']
\end{tikzcd}$$

Also, r_f is the unique real continuous function on τ such that $i \circ r_f = f \circ j$.

(2) *If $r_f = r_g$, then $f = g$.*

(3) $r_{(f \diamond g)} = r_f \diamond r_g$, *for every* $\diamond \in \{+, ., \wedge, \vee\}$.

Proof. (1). For every $p, q \in \mathbb{Q}$, we have

$$\begin{aligned}
(i \circ r_f)(p, q) &= (i \circ f \circ j)(p, q) \\
&= (i \circ f)(]\!]p, q[\![) \\
&= i(f(]\!]p, q[\![)) \\
&= f(]\!]p, q[\![) \\
&= (f \circ j)(p, q).
\end{aligned}$$

Therefore, $i \circ r_f = f \circ j$. The proof of uniqueness is trivial. (2). For every $r \in \mathbb{R}$, we have

$$\begin{aligned}
f(\{r\}) &= f(\bigcap\{]\!]p, q[\![\,|\, p, q \in \mathbb{Q}, p < r < q\}) \\
&= \bigwedge\{f(]\!]p, q[\![) \,|\, p, q \in \mathbb{Q}, p < r < q\} && \text{by Proposition 4.2} \\
&= \bigwedge\{(f \circ j)(p, q) \,|\, p, q \in \mathbb{Q}, p < r < q\} \\
&= \bigwedge\{r_f(p, q) \,|\, p, q \in \mathbb{Q}, p < r < q\} \\
&= \bigwedge\{r_g(p, q) \,|\, p, q \in \mathbb{Q}, p < r < q\} \\
&= \bigwedge\{(g \circ j)(p, q) \,|\, p, q \in \mathbb{Q}, p < r < q\} \\
&= \bigwedge\{g(]\!]p, q[\![) \,|\, p, q \in \mathbb{Q}, p < r < q\} \\
&= g(\bigcap\{]\!]p, q[\![\,|\, p, q \in \mathbb{Q}, p < r < q\}) && \text{by Proposition 4.2} \\
&= g(\{r\}).
\end{aligned}$$

Hence $f = g$, by Lemma 2.2.

(3). By Lemma 2.2, we have

$$\begin{aligned} r_{f \diamond g}(p,q) &= (f \diamond g) \circ j(p,q) \\ &= (f \diamond g)(\rrbracket p,q \llbracket) \\ &= \bigvee\{(f(\rrbracket r,s\llbracket) \wedge g(\rrbracket u,v\llbracket) : \rrbracket r,s\llbracket \diamond \rrbracket u,v\llbracket \subseteq \rrbracket p,q\llbracket\} \\ &= \bigvee\{(f \circ j)(r,s) \wedge (g \circ j)(u,v) : \rrbracket r,s\llbracket \diamond \rrbracket u,v\llbracket \subseteq \rrbracket p,q\llbracket\} \\ &= \bigvee\{r_f(r,s) \wedge r_g(u,v) : < r,s > \diamond < u,v > \subseteq < p,q >\} \\ &= (r_f \diamond r_g)(p,q), \end{aligned}$$

for every $p, q \in \mathbb{Q}$. Therefore, $r_{f \diamond g} = r_f \diamond r_g$. $\square$

In [15], we show that $F(L)$ is isomorphic to a sub-f-ring of $\mathcal{R}L$ and consequently $\mathcal{R}L_\tau$ is too. In [3], Banaschewski proves that if $h : M \longrightarrow L$ is a dense frame homomorphism, then the f-ring homomorphism $Rh : \mathcal{R}M \longrightarrow \mathcal{R}L$ is injective, which then makes $\mathcal{R}M$ isomorphic to some sub-f-ring of $\mathcal{R}L$. Note that the following theorem has also a good idea to extract the most interesting sub-ring $\mathcal{R}L_\tau$ from $\mathcal{R}\tau$.

Theorem 4.4. *$\mathcal{R}L_\tau$ is isomorphic to a sub-f-ring of $\mathcal{R}\tau$.*

Proof. By Lemma 4.3, the map $r : \mathcal{R}L_\tau \longrightarrow \mathcal{R}\tau$ given by $r(f) = r_f$ is an f-ring monomorphism as desired. $\square$

Corollary 4.5. *The f-ring $\mathcal{R}L_\tau$ is a semiprime and archimedean ring.*

Proof. $\mathcal{R}\tau$ is semiprime and archimedean (see [4]), and consequently, so is $\mathcal{R}L_\tau$, by Theorem 4.4. $\square$

An immediate consequence of this last theorem is that for every $f, g \in \mathcal{R}(L_\tau)$, the following statements are equivalent:

(1) $f \leq g$.

(2) $r_f \leq r_g$.

(3) For every $p \in \mathbb{Q}$, $f(p, +\infty) \leq g(p, +\infty)$.

(4) For every $q \in \mathbb{Q}$, $f(-\infty, q) \geq g(-\infty, q)$.

5 Boolean algebras

In this section, we show that if L_τ is a topoframe, then there exists a complete Boolean algebra B such that τ is a topoframe on B and $\mathcal{R}L_\tau$ is isomorphic to a sub- f-ring of $\mathcal{R}(B_\tau)$.

It is well-known that if L is a pseudocomplemented distributive lattice, then

$$\mathfrak{B}L := \{a \in L \mid a^{**} = a\}$$

with $a \vee^{\mathfrak{B}L} b := (a \vee^L b)^{**}$ and $a \wedge^{\mathfrak{B}L} b := a \wedge^L b$, for every $a, b \in \mathfrak{B}L$, is a Boolean algebra. Also for a frame L, the subset $\mathfrak{B}L$ is a complete Boolean algebra, with the meet as in L and join $(\bigvee_i^L a_i)^{**}$, was known as Booleanization of L and in localic language, it is the smallest dense sublocale of L. Nonetheless, we look now at the weaker conditions on L to make Booleanization.

Lemma 5.1. *If L is a pseudocomplemented distributive lattice, then for every $\{a_i\}_{i\in I} \subseteq \mathfrak{B}L$ such that the supremum $\{a_i\}_{i\in I}$ exists in L, we have*

$$\bigvee_i^{\mathfrak{B}L} a_i = (\bigvee_i^{L} a_i)^{**},$$

and hence if L is a frame, then $\mathfrak{B}L$ is a complete Boolean algebra.

Proof. First, note that $\leq^{\mathfrak{B}L} = \leq^L$, because $\wedge^{\mathfrak{B}L} = \wedge^L$. Let $\{a_i\}_{i\in I} \subseteq \mathfrak{B}L$, then we have

$$\begin{aligned} a_i \leq^L \textstyle\bigvee_i^L a_i \quad &\Rightarrow \quad (\textstyle\bigvee_i^L a_i)^* \leq^L a_i^* \\ &\Rightarrow \quad a_i = a_i^{**} \leq^L (\textstyle\bigvee_i^L a_i)^{**} \\ &\Rightarrow \quad a_i \leq^{\mathfrak{B}L} (\textstyle\bigvee_i^L a_i)^{**} \end{aligned}$$

for every $i \in I$. Now, we assume that $c \in \mathfrak{B}L$ is an upper bound for $\{a_i\}_{i\in I}$. Hence, $a_i \wedge^L c = a_i \wedge^{\mathfrak{B}L} c = a_i$, that is $a_i \leq^L c$ for every $i \in I$. So that we have

$$\begin{aligned} \textstyle\bigvee_i^L a_i \leq^L c \quad &\Rightarrow \quad c^* \leq^L (\textstyle\bigvee_i^L a_i)^* \\ &\Rightarrow \quad (\textstyle\bigvee_i^L a_i)^{**} \leq^L c^{**} = c \\ &\Rightarrow \quad (\textstyle\bigvee_i^L a_i)^{**} \leq^{\mathfrak{B}L} c. \end{aligned}$$

Therefore, $\bigvee_i^{\mathfrak{B}L} a_i = (\bigvee_i^L a_i)^{**}$. ∎

Proposition 5.2. *Let L be a frame and N a subframe of L. Then*

$$N^{**} := \{n^{**} : n \in N\}$$

is a topoframe on $\mathfrak{B}L$, where $n^* = \bigvee\{x \in L : x \wedge n = \bot\}$.

Proof. Define the mapping $\theta : L \longrightarrow \mathfrak{B}L$ by $\theta(a) = a^{**}$ for any $a \in L$. Let $\{a_i\}_i \subseteq L$. Then, by Proposition 2.1, we have

$$\bigwedge_i a_i^{***} = \bigwedge_i a_i{}^* \text{ iff } (\bigvee_i a_i^{**})^* = (\bigvee_i a_i)^* \text{ iff } (\bigvee_i a_i^{**})^{**} = (\bigvee_i a_i)^{**}.$$

Therefore,

$$\theta(\bigvee_i a_i) = (\bigvee_i a_i)^{**} = (\bigvee_i a_i^{**})^{**} = \bigvee_i^{\mathfrak{B}L} a_i^{**} = \bigvee_i^{\mathfrak{B}L} \theta(a_i)\,.$$

Moreover, for every $a, b \in L$,

$$\theta(a \wedge b) = (a \wedge b)^{**} = a^{**} \wedge b^{**} = a^{**} \wedge^{\mathfrak{B}L} b^{**} = \theta(a) \wedge^{\mathfrak{B}L} \theta(b)\,.$$

Hence θ is a frame map. Clearly, $N^{**} := \theta(N)$ be a subframe of $\mathfrak{B}L$. Since $\mathfrak{B}L$ is a complete Boolean algebra, N^{**} is a topoframe on $\mathfrak{B}L$. □

Let L be a frame and M be a completion of L. Then L^{**} is a topoframe on $\mathfrak{B}M$. It is worth looking at another example. Let $f : \mathcal{P}(\mathbb{R}) \to L$ is a frame map. Then $f(L)$, the image of L under f, is in fact a subframe of L. So we conclude that $f(L)$ is a topoframe on $\mathfrak{B}L$. Marvelously, $f(L)$ forms a complete Boolean algebra not only by meet and join of L, but also by meet and join of $\mathfrak{B}L$.

Corollary 5.3. *Let L be a frame.*

(1) *Let N be a subframe of L. For any $a, b \in N$, if $a^* = b^*$ implies $a = b$, then N is an isomorphism to a subframe of $\mathfrak{B}L$.*

(2) *let τ be a topoframe on L. Then τ is a topoframe on $\mathfrak{B}L$.*

(3) *L is a complete Boolean algebra if and only if for every $a, b \in L$, $a^* = b^*$ implies $a = b$.*

Proof. (1). By Proposition 5.2, it suffices to prove that the restriction of θ to N is one-to-one. For, let $a, b \in N$ such that $a^{**} = b^{**}$. So that $a^* = b^*$ and then $a = b$, by hypothesis.

(2). By part (1), $\tau = \tau^{**}$ is a subframe of $\mathfrak{B}L$, and since all elements of τ are complementary elements in L and also in $\mathfrak{B}L$, we infer that τ be a topoframe on $\mathfrak{B}L$.

(3). One direction of this equivalence is obvious. To prove the converse direction, by Proposition 5.2, it suffices to prove that $\theta : L \longrightarrow \mathfrak{B}L$ with $\theta(a) = a^{**}$ is bijective. Note that BL is also equal to the set $\{a^{**} : a \in L\}$; and hence θ is a onto map. To prove that θ is a one-to-one map, use an argument similar to the proof of part (1). Hence θ is an isomorphism, by hypothesis. $\square$

The connection between $\mathcal{R}L_\tau$ and $\mathcal{R}(\mathfrak{B}L)_\tau$ given more generally in the following thereom. It is worth mentioning that if $f \in \mathcal{R}L_\tau$, the composition $f^{**} = \theta \circ f$ is clearly in $\mathcal{R}(\mathfrak{B}L)_\tau$, where θ is the frame map introduce in Proposition 5.2.

Theorem 5.4. *Let τ be a topoframe on a frame L. Then, the mapping*

$$\begin{array}{rcl} \varphi : \mathcal{R}L_\tau & \longrightarrow & \mathcal{R}(\mathfrak{B}L)_\tau \\ f & \longmapsto & f^{**} \end{array}$$

is an f-ring monomorphism.

Proof. By definition of $\mathfrak{B}L$, if $A, B \in \mathcal{P}(\mathbb{R})$, then

$$(f(A \cap B))^{**} = (f(A) \wedge f(B))^{**} = f(A)^{**} \wedge f(B)^{**} = \varphi(f)(A) \wedge^{\mathfrak{B}L} \varphi(f)(B).$$

Also, if $\{A_\lambda\}_{\lambda\in\Lambda} \subseteq \mathcal{P}(\mathbb{R})$, then, by Lemma 5.1,

$$f^{**}(\bigcup_{\lambda\in\Lambda} A_\lambda) = (f(\bigcup_{\lambda\in\Lambda} A_\lambda))^{**} = (\bigvee_{\lambda\in\Lambda}^{L} f(A_\lambda))^{**} = \bigvee_{\lambda\in\Lambda}^{\mathfrak{B}L} f(A_\lambda).$$

Hence $f^{**} : \mathcal{P}(\mathbb{R}) \to \mathfrak{B}L$ is a frame map, and by Corollary 5.3, f^{**} is real-continuous.

If $f, g \in \mathcal{R}L_\tau$ and $\varphi(f) = \varphi(g)$, then

$$f(A) = f(A)^{**} = \varphi(f)(A) = \varphi(g)(A) = g(A)^{**} = g(A)$$

for every $A \in \mathcal{P}(\mathbb{R})$. So $f = g$ and hence φ is a one-one frame map.

If $f, g \in \mathcal{R}L_\tau$ and $A \in \mathcal{P}(\mathbb{R})$, then

$$\begin{aligned}
\varphi(f \diamond g)(A) &= (f \diamond g)^{**}(A) \\
&= ((f \diamond g)(A))^{**} \\
&= (\textstyle\bigvee^{L}\{f(\{x\}) \wedge g(\{y\}) : x \diamond y \in A\})^{**} \\
&= \textstyle\bigvee^{\mathfrak{B}L}\{f(\{x\}) \wedge g(\{y\}) : x \diamond y \in A\} \qquad \text{by Lemma 5.1} \\
&= \textstyle\bigvee^{\mathfrak{B}L}\{f(\{x\}) \wedge^{\mathfrak{B}L} g(\{y\}) : x \diamond y \in A\} \\
&= \textstyle\bigvee^{\mathfrak{B}L}\{(f(\{x\}))^{**} \wedge^{\mathfrak{B}L} (g(\{y\}))^{**} : x \diamond y \in A\} \\
&= \textstyle\bigvee^{\mathfrak{B}L}\{\varphi(f)(\{x\}) \wedge^{\mathfrak{B}L} \varphi(g)(\{y\}) : x \diamond y \in A\} \\
&= (\varphi(f) \diamond \varphi(g))(A)
\end{aligned}$$

for every $\diamond \in \{+, ., \wedge, \vee\}$. Therefore, φ is an f-ring embedding. $\square$

Remark 5.5. Given a topoframe L_τ, let $M = \langle \tau \cup \tau' \rangle$, the subframe of L generated by $\tau \cup \tau'$. Then, every $\tau-$real continuous function $f : \mathcal{P}(\mathbb{R}) \to L$ factors through M, since $f(X) = \bigvee_{x \in X}^{L} f(\{x\}) \in \langle \tau' \rangle \subseteq M$ for every $X \subseteq \mathbb{R}$. It is clearly still a $\tau-$real continuous function, when we consider it as taking values in M. So $\mathcal{R}L_\tau = \mathcal{R}M_\tau$, and we can embed $\mathcal{R}L_\tau$ into $\mathcal{R}(\mathfrak{B}M)_\tau$.

6 L-extendable real continuous functions

The problem that "when $\mathcal{R}L_\tau$ is isomorphic to $\mathcal{R}\tau$?" is not still solved. If the mapping r given in Theorem 4.4 were onto, we'd be able to replace any $\mathcal{R}\tau$ by $\mathcal{R}L_\tau$ for some frame L. However, our attempt to do this, caused to consider the extendability of a real continuous function on a frame to a real continuous function on a topoframe, as the following related results shows.

Definition 6.1. Let L be a frame. A real-trail on L is a map $t : \mathbb{R} \longrightarrow L$ such that

1. $\bigvee_{x \in \mathbb{R}} t(x) = \top$,
2. $t(x) \wedge t(y) = \bot$ for any $x, y \in \mathbb{R}$ with $x \neq y$.

Lemma 6.2. *For any real-trail t on a frame L,*

$$\begin{array}{rcl} \varphi : P(\mathbb{R}) & \longrightarrow & L \\ X & \longmapsto & \bigvee_{x\in X} t(x) \end{array}$$

is a frame map.

Proof. We check the conditions of the frame map for φ. It is clear that $\varphi(\emptyset) = \bot$. By condition (1) of Definition 6.1,

$$\varphi(\mathbb{R}) = \bigvee_{x\in\mathbb{R}} t(x) = \top\,.$$

Let $X, Y \in P(\mathbb{R})$. Then

$$\begin{array}{rll} \varphi(X) \wedge \varphi(Y) & = \bigvee_{x\in X} t(x) \wedge \bigvee_{y\in Y} t(y) & \\ & = \bigvee_{x\in X, y\in Y} (t(x) \wedge t(y)) & \\ & = \bigvee_{r\in X\cap Y} t(r) & \text{Definition 6.1(2)} \\ & = \varphi(X \cap Y). & \end{array}$$

For every $\{X_i\}_i \subseteq P(\mathbb{R})$, we have

$$\begin{array}{rl} \varphi(\bigcup_i X_i) & = \bigvee_{x\in\bigcup_i X_i} t(x) \\ & = \bigvee_i \bigvee_{r\in X_i} t(x) \\ & = \bigvee_i \varphi(X_i). \end{array}$$

This completes the proof. $\square$

Lemma 6.3. *Let M be a frame and let L be a regular frame. If $f, g : L \longrightarrow M$ are frame morphisms such that for every $a \in L$, $f(a) \leq g(a)$, then $f = g$.*

Proof. The frame L is a regular, so $a = \bigvee_{t\prec a} t$, and hence

$$g(a) = \bigvee_{t\prec a} g(t) \leq f(a),$$

because

$$\begin{aligned}
t \prec a \quad &\Rightarrow \quad t^* \vee a = \top \\
&\Rightarrow \quad f(t^*) \vee f(a) = f(\top) = \top \\
&\Rightarrow \quad g(t^*) \vee f(a) = \top \qquad\qquad \text{since } f(t^*) \leq g(t^*) \\
&\Rightarrow \quad g(t)^* \vee f(a) = \top \qquad\qquad \text{since } g(t^*) \leq (g(t))^* \\
&\Rightarrow \quad g(t) \prec f(a) \\
&\Rightarrow \quad g(t) \leq f(a).
\end{aligned}$$

□

Definition 6.4. Let τ be a topoframe on a frame L. The frame map $\alpha \in \mathcal{R}\tau$ is called L-*extendable* real continuous function if and only if for every $r \in \mathbb{R}$,

$$\bigvee_{r\in\mathbb{R}}^{L} (\alpha(-,r) \vee \alpha(r,-))' = \top\,,$$

where $(-,r) = \bigvee_{\substack{s\in\mathcal{Q} \\ s\lneqq r}}(-,s)$ and $(r,-) = \bigvee_{\substack{s\in\mathcal{Q} \\ r\lneqq s}}(s,-)$.

Example 6.5. For every $f \in \mathcal{R}L_\tau$, the mapping r_f given at the beginning of this section is L-extendable, because

$$\begin{aligned}
\textstyle\bigvee_{r\in\mathbb{R}}^{L}(r_f(-,r) \vee r_f(r,-))' &= \textstyle\bigvee_{r\in\mathbb{R}}^{L}(f]\!]-,r[\![\vee f]\!]r,-[\![)' \\
&= \textstyle\bigvee_{r\in\mathbb{R}}^{L}(f(\mathbb{R} - \{r\})' \\
&= \textstyle\bigvee_{r\in\mathbb{R}}^{L}(f(\{r\}) \\
&= \top\,.
\end{aligned}$$

Let L_τ be a topoframe. The set of all L-extendable real continuous functions of $\mathcal{R}\tau$ denoted by $\mathcal{R}^L(\tau)$.

Proposition 6.6. *Let L_τ be a topoframe. For every L-extendable map $\alpha \in \mathcal{R}\tau$, the following properties hold.*

(1) *The mapping e_α, defined by*

$$e_\alpha(S) = \bigvee_{x\in S}^{L} (\alpha(-,x) \vee \alpha(x,-))' \qquad (S \in \mathcal{P}(\mathbb{R})),$$

is a frame homomorphism of $\mathcal{P}(\mathbb{R})$ into L.

(2) *Let i and j be functions given in the beginning of this section, then the following diagram commutes.*

$$\begin{tikzcd} P(\mathbb{R}) \arrow[r, "e_\alpha"] & L \\ \mathcal{L}(\mathbb{R}) \arrow[u, "j"] \arrow[r, "\alpha"] & \tau \arrow[u, "i"'] \end{tikzcd}$$

The frame map e_α is an $(\mathfrak{O}(\mathbb{R}), \tau)$-homomorphism. Also, e_α is the unique real continuous function on L_τ such that $e_\alpha \circ j = i \circ \alpha$. Moreover, e_α can be redefined by

$$e_\alpha(S) = \bigvee_{s \in S} \bigwedge \{\alpha(p,q) \mid p < s < q,\ p,q \in \mathbb{Q}\}\,.$$

(3) *For every $\beta, \gamma \in \mathcal{R}^L(\tau)$, if $e_\beta = e_\gamma$, then $\beta = \gamma$.*

Proof. (1). For any $x \in \mathbb{R}$,

$$t_\alpha(x) := (\alpha(-,x) \vee \alpha(x,-))'$$

is a real-trail on L, since $\bigvee^L_{x \in \mathbb{R}} t_\alpha(x) = \top$, by hypothesis, and for any $x, y \in \mathbb{R}$ such that $x \neq y$,

$$\begin{aligned} t_\alpha(x) \wedge t_\alpha(y) &= (\alpha(-,x) \vee \alpha(x,-))' \wedge (\alpha(-,y) \vee \alpha(y,-))' \\ &= (\alpha(-,x) \vee \alpha(x,-) \vee (\alpha(-,y) \vee \alpha(y,-))' \\ &= (\top)' \qquad \text{since } x \neq y \text{ and } \alpha \text{ preserves suprema} \\ &= \bot\,. \end{aligned}$$

Hence $e_\alpha(S) = \bigvee^L_{x \in S} t_\alpha(x)$ is a frame map, by Lemma 6.2.

(2). For every $p, q \in \mathbb{Q}$, we have

$$\begin{aligned} (e_\alpha \circ j)((p,q)) &= e_\alpha(]\!]p,q[\![) \\ &= \bigvee\nolimits^L_{\substack{x \in \mathbb{R} \\ p \lneqq x \lneqq q}} (\alpha(-,x) \vee \alpha(x,-))' \\ &\leq \alpha(p,q) \\ &= i \circ \alpha(p,q)\,. \end{aligned}$$

Since $\mathfrak{O}(\mathbb{R})$ is a regular frame and $e_\alpha \circ j$ and $i \circ \alpha$ are frame maps, by Lemma 6.3, we conclude that $e_\alpha \circ j = i \circ \alpha$.

To prove that $e_\alpha \in \mathcal{R}L_\tau$, it suffices to show that $e_\alpha(\rrbracket p, q\llbracket) \in \tau$, for every $p, q \in \mathbb{Q}$. If $p, q \in \mathbb{Q}$, then we have

$$\begin{aligned} e_\alpha(\rrbracket p, q\llbracket) &= (e_\alpha \circ j)((p, q)) \\ &= (i \circ \alpha)((p, q)) \\ &= \alpha(p, q) \in \tau. \end{aligned}$$

Let $f : \mathcal{P}(\mathbb{R}) \longrightarrow L$ be a $(\mathfrak{O}(\mathbb{R}), \tau)$-homomorphism such that $f \circ j = i \circ \alpha$, then for every $S \in \mathcal{P}(\mathbb{R})$, using Proposition 4.2,

$$\begin{aligned} f(S) &= f(\textstyle\bigcup_{s \in S}\{s\}) \\ &= \textstyle\bigvee_{s \in S} f(\{s\}) \\ &= \textstyle\bigvee_{s \in S} f(\{\bigwedge\{\rrbracket p, q\llbracket \mid p < s < q,\, p, q \in \mathbb{Q}\}) \\ &= \textstyle\bigvee_{s \in S} \bigwedge\{f(\rrbracket p, q\llbracket) \mid p < s < q,\, p, q \in \mathbb{Q}\} \\ &= \textstyle\bigvee_{s \in S} \bigwedge\{(f \circ j)(p, q) \mid p < s < q,\, p, q \in \mathbb{Q}\} \\ &= \textstyle\bigvee_{s \in S} \bigwedge\{(i \circ \alpha)(p, q) \mid p < s < q,\, p, q \in \mathbb{Q}\} \end{aligned}$$

$$\begin{aligned} &= \textstyle\bigvee_{s \in S} \bigwedge\{(e_\alpha \circ j)(p, q) \mid p < s < q,\, p, q \in \mathbb{Q}\} \\ &= \textstyle\bigvee_{s \in S} \bigwedge\{e_\alpha(\rrbracket p, q\llbracket) \mid p < s < q,\, p, q \in \mathbb{Q}\} \\ &= \textstyle\bigvee_{s \in S} e_\alpha(\bigwedge\{\rrbracket p, q\llbracket \mid p < s < q,\, p, q \in \mathbb{Q}\}) \\ &= \textstyle\bigvee_{s \in S} e_\alpha(\{s\}) \\ &= e_\alpha(\textstyle\bigcup_{s \in S}\{s\}) \\ &= e_\alpha(S) \end{aligned}$$

to give e_α is unique as required.

(3). Let $e_\gamma = e_\beta$. Then

$$\begin{aligned}
\alpha(p,q) &= (i\circ\alpha)((p,q)) \\
&= (e_\alpha \circ j)((p,q)) \\
&= e_\alpha(\rrbracket p,q\llbracket) \\
&= e_\beta(\rrbracket p,q\llbracket) \\
&= (e_\beta \circ j)((p,q)) \\
&= (i\circ\beta)((p,q)) \\
&= \beta(p,q)
\end{aligned}$$

for every $p,q\in\mathbb{Q}$. This completes the proof of assertion (4). □

Lemma 6.7. *Let L_τ be a topoframe. For every $\alpha,\beta\in\mathcal{R}^L\tau$, and $\diamond\in\{+,.,\vee,\wedge\}$, $e_{\alpha\diamond\beta} = e_\alpha\diamond e_\beta$. Hence $\alpha\diamond\beta\in\mathcal{R}^L\tau$.*

Proof. For any $f\in\mathcal{R}L_\tau$, let $r(f) = f\circ j$. By uniqueness of α in the following commutative diagram (see Lemma 4.3), we have $r(e_\alpha)=\alpha$ (and $r(e_\beta)=\beta$).

$$\begin{tikzcd}
P(\mathbb{R}) \arrow[r, "e_\alpha"] & L \\
\mathcal{L}(\mathbb{R}) \arrow[u, "j"] \arrow[r, "\alpha"] & \tau \arrow[u, "\mathrm{i}"']
\end{tikzcd}$$

So $r(e_\alpha\diamond e_\beta) = r(e_\alpha)\diamond r(e_\beta) = \alpha\diamond\beta$. Also, by Example 6.5, $r(e_\alpha\diamond e_\beta)$ is L-extendable, whence $e_{r(e_\alpha\diamond e_\beta)} = e_{\alpha\diamond\beta}$. So that $e_\alpha\diamond e_\beta = e_{\alpha\diamond\beta}$. □

Theorem 6.8. *Let L_τ be a topoframe. Then $\mathcal{R}^L\tau$ is a sub-f-ring of $\mathcal{R}\tau$.*

Proof. By Lemma 6.7, this is obvious. □

Theorem 6.9. *For any topoframe L_τ, the mapping $\varphi:\mathcal{R}^L\tau\longrightarrow\mathcal{R}L_\tau$ taking any α to e_α is an f-ring isomorphism, where e_α is the function discribed in Proposition 6.6.*

Proof. By Lemma 6.7, φ is an f-ring monomorphism. By Example 6.5, r_f is L-extendable and $\varphi(r_f) = f$, for every $f \in \mathcal{R}L_\tau$. Hence φ is an f-ring isomorphism. $\square$

Acknowledgments

We thank the referee for a thorough scrutiny on the first version of this paper, and for the comments which have improved this work. We also express our gratitude to Professor M. Mehdi Ebrahimi.

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