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Steps toward the weak higher category of weak higher categories in the globular setting

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I dedicate this work to Myles Tierney.

Abstract. We start this article by rebuilding higher operads of weak higher transformations, and correct those in [7]. As in [7] we propose an operadic approach for weak higher n-transformations, for each $n \in \mathbb{N}$, where such weak higher n-transformations are seen as algebras for specific contractible higher operads. The last chapter of this article asserts that, up to precise hypotheses, the higher operad B^0_C of Batanin and the terminal higher operad $B^0_{S_u}$, both have the fractal property. In other words we isolate the precise technical difficulties behind a major problem in globular higher category theory, namely, that of proving the existence of the globular weak higher category of globular weak higher categories.

Introduction

This article is the third in a series of three articles (see [10, 11]). In [7] we have proposed a higher operadic definition of the weak higher transformations which were supposed to be the natural continuity of the work of Michael Batanin after his work on weak higher categories. However,

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André Joyal had pointed out to us that these higher operads contain too much coherences, and the consequence of it is we can find simple examples of natural transformations which are not algebras for these operads, which were supposed to produce all kinds of transformations. The kind of contractibility we used, that is, controlled by the 2-colored globular set of arities $\mathbb{T}(1)+\mathbb{T}(1)$[1], didn't give us a good control of the coherences of this approach of weak higher transformations. We tried to use another kind of the notion of contractibility in [8], but Jacques Penon had pointed out to us that this notion does not produce enough contractions for a correct approach of weak higher transformations: In [7], the notion of contractibility produces too much contractions, and in [8] the notion of contractibility does not have enough contractions. Despite of our promising combinatorics using our *operation systems*[2], we didn't found an adapted notion of contractibility for a good higher operadic approach of weak higher transformations.

In this article, we correct this imperfection, and believe we describe here the correct universal contractible higher operads for all weak higher transformations, thanks to an idea coming from the article [6]. As a matter of fact, in [6], the author builds monads on the category $\mathbb{G}lob^2$ whose algebras are models of all kind of weak higher transformations, and the technology involved in it, is those of n-categorical stretchings (for each $n\in\mathbb{N}$), which are objects that basically use the *strict world* to control the *weak world*, and this is exactly the kind of technology we use here: In this article we use the monad of strict higher functors, and monads of all kinds of strict higher transformations, to control arities of all kinds of operations related to higher transformations. Thus instead of using only the category of ω^0-operads, we use also all ω^n-operads ($n\in\mathbb{N}^*$) where ω^n denotes the monad of the strict n-transformations[3] on $\mathbb{G}lob^2$.

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[1] $\mathbb{T}$ denotes the monad of strict higher categories in [7, 9–11] but in this article we denote this monad by ω^0 (see 2). Also, for each $n\geq 1$, we denote by ω^n the monad on $\mathbb{G}lob^2$ of the strict n-transformations. These notations were suggested for the author by Jacques Penon.

[2] Promising, because, thanks to these operation systems (called in this article C^n-systems for all $n\in\mathbb{N}$), we are able to build many interresting universal 2-colored higher operads of all higher transformations for many kind of *higher structures*, especially those of reflexive ∞-magmas, which the corresponding category contains weak ∞-categories as special objects. See [11].

[3] However, it is important to notice that when contractibility is not involved, the 2-colored globular set of arities $\omega^0(1)+\omega^0(1)$ allows to build many interesting 2-colored

Then we obtain a coglobular object of contractible higher operads (see Section 5) in a category $\mathcal{C}$ which must have small pushouts. This coglobular object is formed by higher operads of all kinds of weak higher transformations. Thanks to Proposition 7.2 in [2], it provides a higher operad over the monad ω^0 of the strict higher categories, called the *Coendomorphism higher operad* associated to this coglobular object. In Section 5, we conjecture that the Batanin operad B^0_C of weak higher categories is *fractal.* If this conjecture is true, then it shows that all weak higher categories in Batanin's sense plus all kinds of weak higher transformations build in our article, form themselves a weak higher category in Batanin's sense. In order to state properly this conjecture, we recall the definition of the *standard action associated to a coglobular operadic object*, of an *algebraic coglobular object of higher operads* and of *fractal higher operads*, which are slight generalization of those introduced in [10].

Surprisingly, with our technology, the case of the *strict* higher category of the *strict* higher categories remains conjectural and is at the same level of difficulty as the weak case. This important fact does not affect at all our technology because, in our framework, the operad for strict higher categories must be seen as a higher operad equipped with a strict version of contractibility with chosen contractible units. For instance it is not difficult to show that the terminal higher operad is the free strictly contractible higher operad with contractible units, generated by the pointed collection[4] C^0. Thus, in our framework, the operad of strict higher categories is an initial object in the category of higher operads equipped with strict contractions, which have contractible units and which are equipped with a C^0-system. See the author's thesis [9].

The plan of this article is as follows.

Section 2 describes, for all $n > 1$, the monads ω^n of strict n-transformations, but also the monad ω^1 of strict higher functors, and the monad ω^0 of strict higher categories. Thanks to the globular category of strict higher transformations, we are able to build the coglobular object $(\omega^\bullet, \kappa, \delta)$ of monads for strict higher transformations. If we apply this coglobular object to the terminal object $(1,1)$ of $\mathbb{G}lob^2$ we obtain the coglobular object $(\omega^\bullet(1,1), \kappa(1,1), \delta(1,1))$ of all arities for operads of strict and weak higher

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higher operads: Such 2-colored higher operads are described in [11].

[4] C^0 is the composition system used by Batanin in [2].

transformations. It is interesting to notice that $(\omega^{\bullet}(1,1),\kappa(1,1),\delta(1,1))$ could be thought of the underlying coglobular set of the free strict higher category of free strict higher categories on $(1,1)$.

Section 3 describes, for $n \geq 1$, combinatorics of the object $C^n = (C_0^n, C_1^n)$ in $\mathbb{G}lob^2$ which are pointed collections for higher transformations. These combinatorics are the same as those described in [7, 9–11] except that the old version of C^n are globular sets instead: Colors 1 and 2 in [7, 9–11] are replaced by pairs of globular sets, as in the article [6]. Then, for each $n \geq 1$, we describe all pointed ω^n-collections $(C^n, a^n, c^n; p^n)$. These pointed ω^n-collections $(C^n, a^n, c^n; p^n)$ contain all basic operations we need to generate higher operad (B_C^n, a^n, c^n) for weak n-transformations of Section 4. We finish this section by describing the coglobular object $(C^{\bullet}, a^{\bullet}, c^{\bullet}; p^{\bullet})$ of all pointed ω^n-collections. It is a coglobular object of a category $\omega^{\bullet}$-$\mathbb{C}oll_p$ which is a kind of "fibred category" whose fibers are the well known monoidal categories ω^n-$\mathbb{C}oll_p$.

Section 4 describes, for $n \geq 1$, the operads (B_C^n, a^n, c^n) for weak n-transformations. We start by describing contractible pointed ω^n-collections which is completely similar to the notion of contractibility described by Michael Batanin in [2]. Then we use a result of Max Kelly in [13] to generate, for all $n \geq 1$, the free contractible higher operad (B_C^n, a^n, c^n) on the pointed ω^n-collection $(C^n, a^n, c^n; p^n)$ described in Section 3. This higher operad B_C^n (for each $n \geq 1$) is the initial object of the category $C^nC\omega^n$-$\mathbb{O}per$ of contractible ω^n-operads equipped with a C^n-system. If $n = 1$, (B_C^1, a^1, c^1) is the higher operad for weak higher functors. If $n > 1$, (B_C^n, a^n, c^n) is the higher operad for weak higher n-transformations. We describe accurately multiplications of these operads in order to clarify possible explicit computations and also "shapes" of B_C^n-algebras. We finish this section with three propositions which state that in dimension 2, B_C^1-algebras are pseudo-2-functors, B_C^2-algebras are pseudo-2-natural transformations, and B_C^3-algebras are modifications. Proofs of these propositions are completely similar to those in [6, 7], thus we prefer to avoid to give it again, because it uselessly make the text longer.

In Section 5 we recall the definition of the standard action associated to a coglobular operadic object, of an algebraic coglobular object of higher operads, and the definition of fractal higher operads. These definition are a slight generalization of those given in [10] where the spirit of examples given in [11] remains unchanged. Then we describe the coglobular object $(B^{\bullet}_{C},\delta,\kappa)$ of operads for weak higher transformations in a category $\omega^{\bullet}$-$\mathbb{O}per$ which is a kind of "fibred category" whose fibers are the categories ω^{n}-$\mathbb{O}per$. This construction is possible, thanks to a result of Tom Leinster in [14]. However, the author has not yet found a nice category $\mathcal{C}$ of higher operads which has pushouts and contains the coglobular object $(B^{\bullet}_{C},\delta,\kappa)$ in order to build the corresponding operad of coendomorphism $Coend(B^{\bullet}_{C})$. Thus, we suppose that this coglobular object lives in a category[5] $\mathcal{C}$ having small pushouts. Then, we state our conjecture: We believe the operad B^{0}_{C} of Batanin is fractal. A way to prove this property is to prove that $Coend(B^{\bullet}_{C})$ is contractible and equipped with a C^0-system. If our conjecture is true then it will show that the globular weak higher category of globular weak higher categories exists. We finish this article by describing the coglobular object $(B^{\bullet}_{S_u},\delta,\kappa)$ of operads for strict higher transformations in the same category $\omega^{\bullet}$-$\mathbb{O}per$ as above. We present these operads $B^{n}_{S_u}$ $(n\in\mathbb{N})$ similarly to the B^{n}_{C}: for that we use a strict version of contractibility. Then we conjecture that the terminal ω^0-operad $B^{0}_{S_u}$ of strict higher categories is fractal for $(B^{\bullet}_{S_u},\delta,\kappa)$. This conjecture looks bizarre, because it is well known that the strict higher category of strict higher categories exists. However, it shows that our technology provides a unify and precise technical problem to solve these conjectures related to these fractals phenomenons, which are in the same level of difficulty, when some kinds of contractibility are involved.

1 Conventions and abuse of languages

The category of the small strict ∞-categories is denoted by ∞-$\mathbb{C}at$, and ∞-$\mathbb{C}AT$ denotes the category of the strict ∞-categories. $\mathbb{S}et$ denotes the category of small sets, and $\mathbb{S}ET$ denotes the category of sets and large sets. The sketch of the coglobular sets is denoted[6] by $\mathbb{G}^0$. $\mathbb{G}lob$ denotes the category of globular sets, and $\mathbb{G}lob^2$ denotes the product of $\mathbb{G}lob$ with itself in $\mathbb{C}AT$. Sources and targets of globular sets use the letters s and t. The terminal object of $\mathbb{G}lob$ is denoted by 1, and the terminal object of $\mathbb{G}lob^2$

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[5] The author believes that such a category does not deserve to be conjectural, because several such categories are candidates. We prefer to postpone such a nice choice of $\mathcal{C}$ for a future work.

[6] The usual notation is $\mathbb{G}$ and usually called the *globe category.*

is denoted by $(1,1)$. Also $\mathbb{G}LOB$ denotes the category of globular sets and large globular sets. $\mathbb{G}\mathbb{C}at$ denotes the category of small globular categories, and $\mathbb{G}\mathbb{C}AT$ denotes the category of globular categories. If G is a globular set or a large globular set then $G(m)$ denotes its set (or its large set) of m-cells. For all reflexive globular sets, their operations of reflexivity share the same notations 1^{m}_{m+1} for all integers $m \geq 0$. $\mathbb{M}nd_f(\mathbb{G}lob^2)$ denotes the category of finitary monads on $\mathbb{G}lob^2$. Forgetfull functors are often denoted by the letters U or V, and their left adjoints are often denoted by the letters F or H. All coglobular objects $(W^{\bullet},\delta,\kappa)$ are denoted by the same letters δ and κ, and all globular objects $(W^{\bullet},\sigma,\beta)$ are denoted by the same letters σ and β. Also collections and operads share the same notation for their maps of arities denoted by the letter a, and their maps of coarities denoted by the letter c. Finally monads (S,η,μ) share also the same notation η for their universal maps, and μ for their multiplications. Contexts of this article should avoid any confusions. For this article the reader must be aware about basic notions of higher operads as defined in [2] and $\mathbb{T}$-categories as defined in [5, 14].

2 The strict n-transformations

2.1 Monads ω^n of the strict n-Transformations

Definition 2.1. Consider C and C' two objects of ∞-$\mathbb{C}at$. A strict ∞-functor $C \xrightarrow{F} C'$ between C and C' is a morphism of ∞-$\mathbb{C}at$. Thus it is a morphism of globular sets which preserves the strict ∞-structures of C and C', which means that

- If x and y are two m-cells of C such that $y \circ^m_p x$ is defined then $F(y \circ^m_p x) = F(y) \circ^m_p F(x)$
- If x is a p-cell of C and the m-cell $1^p_m(x)$ is the reflexion of x then $F(1^p_m(x)) = 1^p_m(F(x))$

A morphism between two strict ∞-functors $C \xrightarrow{F} C'$ and $D \xrightarrow{G} D'$ is a morphism of the category of arrows $\mathbb{A}rr(\infty\text{-}\mathbb{C}at)$. Thus, it is given by two strict ∞-functors $C \xrightarrow{H} D$ and $C' \xrightarrow{H'} D'$ such that $H'F = F'H$. The category of strict ∞-functors is denoted by ∞-$\mathbb{F}unct$.

We have a monadic forgetful functor $\infty\text{-}\mathbb{F}unct \xrightarrow{U^1} \mathbb{G}lob^2$ whose left adjoint is denoted by F^1. The monad of the strict ∞-functors is denoted by $(\omega^1, \eta^1, \mu^1)$, or ω^1 for short, and is an object of $\mathbb{M}nd_f(\mathbb{G}lob^2)$. Here $1_{\mathbb{G}lob^2} \xrightarrow{\eta^1} \omega^1$ is the universal map of ω^1.

Definition 2.2. Consider two strict ∞-functors F and G between two objects C and D of $\infty\text{-}\mathbb{C}at$

$$\begin{tikzcd} C \arrow[r, shift left, "F"] \arrow[r, shift right, "G"'] & D \end{tikzcd}$$

A strict ∞-natural transformation $F \xrightarrow{\tau} G$ between F and G is a 2-cell in $\infty\text{-}\mathbb{C}at$. More precisely τ is given by a morphism in $\mathbb{S}et$

$$\begin{tikzcd} C(0) \arrow[r, "\tau"] & D(1) \end{tikzcd}$$

such that for all 1-cell $a \xrightarrow{f} b$ of C we have $\tau(b) \circ^1_0 F(f) = G(f) \circ^1_0 \tau(a)$. Consider another strict ∞-natural transformation $F' \xrightarrow{\tau'} G'$ such that: $\begin{tikzcd} C' \arrow[r, shift left, "F'"] \arrow[r, shift right, "G'"'] & D' \end{tikzcd}$. A morphism between two strict ∞-natural transformations $\tau \longrightarrow \tau'$ is given by two strict ∞-functors $C \xrightarrow{H} C'$ and $D \xrightarrow{K} D'$ such that $1^1_2(K) \circ^2_0 \tau = \tau' \circ^2_0 1^1_2(H)$. The category of strict ∞-natural transformations is denoted $(2,\infty)\text{-}\mathbb{T}rans$.

We have a monadic forgetful functor $(2,\infty)\text{-}\mathbb{T}rans \xrightarrow{U^2} \mathbb{G}lob^2$ whose left adjoint is denoted by F^2. The monad of the strict ∞-natural transformations is denoted by $(\omega^2, \eta^2, \mu^2)$, or ω^2 for short, and is an object of $\mathbb{M}nd_f(\mathbb{G}lob^2)$. Here $1_{\mathbb{G}lob^2} \xrightarrow{\eta^2} \omega^2$ is the universal map of ω^2. Let us rename by strict 2-transformations objects of $(2,\infty)\text{-}\mathbb{T}rans$[7]. For integers $n \geq 3$, we define the strict n-transformations by induction.

Definition 2.3. Suppose the categories $(k,\infty)\text{-}\mathbb{T}rans$ of the strict k-transformations are defined for all $k \in [\![2, n-1]\!]$. A strict n-transformation

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[7] With this terminology, the strict ∞-functors could be called the strict 1-transformations and the strict ∞-categories could be called the strict 0-transformations.

$\alpha \xrightarrow{\xi} \beta$ between the strict $(n-1)$-transformations α and β is given by a morphism in $\mathbb{S}et$

$$C(0) \xrightarrow{\quad\xi\quad} D(n)$$

such that for all n-cell a of C with $s_0^n(f) = a$ and $t_0^n(f) = b$ we have $1_n^1(\xi(b)) \circ_0^n F(f) = G(f) \circ_0^n 1_n^1(\xi(a))$. Consider another strict n-transformation $\alpha' \xrightarrow{\xi'} \beta'$ such that $s_0^n(\xi') = C'$ and $t_0^n(\xi') = G'$. A morphism between two strict n-transformations $\xi \longrightarrow \xi'$ is given by two strict ∞-functors $C \xrightarrow{H} C'$ and $D \xrightarrow{K} D'$ such that $1_n^1(K) \circ_0^n \xi = \xi' \circ_0^n 1_n^1(H)$. The category of strict n-transformations is denoted by (n,∞)-$\mathbb{T}rans$.

We have a monadic forgetful functor (n,∞)-$\mathbb{T}rans \xrightarrow{U^n} \mathbb{G}lob^2$ whose left adjoint is denoted by F^n. The monad of the strict n-transformations is denoted by $(\omega^n, \eta^n, \mu^n)$, or ω^n for short, and is an object of $\mathbb{M}nd_f(\mathbb{G}lob^2)$. Here $1_{\mathbb{G}lob^2} \xrightarrow{\eta^n} \omega^n$ is the universal map of ω^n.

Theorem 2.4. *For each $n \geq 1$, the monad $(\omega^n, \eta^n, \mu^n)$ on $\mathbb{G}lob^2$ of the strict n-transformations is cartesian.*

We prove this theorem by using systematicaly the cartesianity of the monad $(\omega^0, \eta^0, \mu^0)$ on $\mathbb{G}lob$ of the strict higher categories. Accurate proof of this theorem shall be described in [12].

In [11] we define a coglobular object in $\mathbb{C}$at

$$\begin{tikzcd}
\mathbb{G}_0^{op} \arrow[r, shift left, "\delta_0^1"] \arrow[r, shift right, "\kappa_0^1"'] & \mathbb{G}_1^{op} \arrow[r, shift left, "\delta_2^1"] \arrow[r, shift right, "\kappa_1^2"'] & \mathbb{G}_2^{op} \arrow[r, dotted] & \mathbb{G}_{n-1}^{op} \arrow[r, shift left, "\delta_{n-1}^n"] \arrow[r, shift right, "\kappa_{n-1}^n"'] & \mathbb{G}_n^{op} \arrow[r, dotted] & {}
\end{tikzcd}$$

such that when we apply to it the contravariant functor $[-;\mathbb{S}et]$ we obtain the globular category of globular sets[8].

$$\begin{tikzcd}
{} \arrow[r, dotted] & {[\mathbb{G}_n^{op};\mathbb{S}et]} \arrow[r, shift left, "\sigma_{n-1}^n"] \arrow[r, shift right, "\beta_{n-1}^n"'] & {[\mathbb{G}_{n-1}^{op};\mathbb{S}et]} \arrow[r, dotted] & {[\mathbb{G}_1^{op};\mathbb{S}et]} \arrow[r, shift left, "\sigma_0^1"] \arrow[r, shift right, "\beta_0^1"'] & {[\mathbb{G}_0^{op};\mathbb{S}et]}
\end{tikzcd}$$

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[8] It is easy to see that if we apply to it the contravariant functor $[-;\mathbb{C}at]$ we obtain the globular category of small globular categories.

An object of the category of presheaves $[\mathbb{G}_n^{op};\mathbb{S}et]$ is called an (n,ω)-graph and morphisms in $[\mathbb{G}_n^{op};\mathbb{S}et]$ are just natural transformations between such presheaves.

Proposition 2.5. *For each $n \geq 2$ the category (n,∞)-$\mathbb{T}rans$ is a full subcategory of $[\mathbb{G}_n^{op};\mathbb{S}et]$. Also the categories ∞-$\mathbb{F}unct$ and ∞-$\mathbb{C}at$ are, respectively, full subcategories of $[\mathbb{G}_1^{op};\mathbb{S}et]$ and $[\mathbb{G}_0^{op};\mathbb{S}et]$. Also sources and targets functors σ^n_{n-1} and β^n_{n-1} of the globular category just above respect these strict structures and their restrictions give the globular category of strict ∞-categories*

$$\begin{tikzcd} \cdots \arrow[r, dotted] & (n,\infty)\text{-}\mathbb{T}rans \arrow[r, shift left, "\sigma^n_{n-1}"] \arrow[r, shift right, "\beta^n_{n-1}"'] & (n-1,\infty)\text{-}\mathbb{T}rans \arrow[r, dotted] & \infty\text{-}\mathbb{F}unct \arrow[r, shift left, "\sigma^1_0"] \arrow[r, shift right, "\beta^1_0"'] & \infty\text{-}\mathbb{C}at \ . \end{tikzcd}$$

Now consider the globe functor $\mathbb{C}AT \xrightarrow{GLOB} \mathbb{G}\mathbb{C}AT$ and the object functor $\mathbb{C}AT \xrightarrow{OBJ} \mathbb{S}ET$. If we apply $GLOB$ on OBJ we obtain the functor $\mathbb{G}\mathbb{C}AT \xrightarrow{GLOB} \mathbb{G}LOB$ which sends a globular category to its object part. If we apply it to the globular category of strict ∞-categories we obtain the following globular object in $\mathbb{S}ET$

$$\begin{tikzcd} \cdots \arrow[r, dotted] & (n,\infty)\text{-}\mathbb{T}rans(0) \arrow[r, shift left, "\sigma^n_{n-1}"] \arrow[r, shift right, "\beta^n_{n-1}"'] & (n-1,\infty)\text{-}\mathbb{T}rans(0) \arrow[r, dotted] & \infty\text{-}\mathbb{F}unct(0) \arrow[r, shift left, "\sigma^1_0"] \arrow[r, shift right, "\beta^1_0"'] & \infty\text{-}\mathbb{C}at(0) \ . \end{tikzcd}$$

It is not difficult to see that this globular set is equipped with a canonical structure of strict ∞-category (see for example [6]). This ∞-categorical structure on ∞-$\mathbb{C}$at is well known, but we need this accurate description of its underlying globular set as above for the sections 2.2 and 5.2.

2.2 The coglobular object of the monads ω^n

In this section we use the category $\mathbb{A}dj$ of adjunctions and the category $\mathbb{M}nd$ of categories equipped with a monad as defined in [6], and we freely use the fact that there is a canonical functor $\mathbb{A}dj \xrightarrow{U} \mathbb{M}nd$ which is left adjoint $U \dashv A$, where A is the Eilenberg-Moore construction.

The globular category $((\bullet,\infty)\text{-}\mathbb{T}rans,\sigma,\beta)$ and the adjunctions

$F^n \dashv U^n (n \in \mathbb{N})$ (see Section 2) allow to build a globular object in $\mathbb{A}dj$:

$$\begin{tikzcd}
\cdots \arrow[r, dotted] & (n,\infty)\text{-}\mathbb{T}rans \arrow[r, shift left, "\sigma^n_{n-1}"] \arrow[r, shift right, "\beta^n_{n-1}"'] & (n-1,\infty)\text{-}\mathbb{T}rans \arrow[r, dotted] & \infty\text{-}\mathbb{F}unct \arrow[r, shift left, "\sigma^1_0"] \arrow[r, shift right, "\beta^1_0"'] & \infty\text{-}\mathbb{C}at \\
\cdots \arrow[r, dotted] & \mathbb{G}lob^2 \arrow[r, shift left] \arrow[r, shift right] \arrow[u, shift left, "F^n"] \arrow[u, phantom, "\dashv"] & \mathbb{G}lob^2 \arrow[r, shift left] \arrow[r, shift right] \arrow[u, shift left, "F^{n-1}"] \arrow[u, phantom, "\dashv"] & \mathbb{G}lob^2 \arrow[r, shift left] \arrow[r, shift right] \arrow[u, shift left, "F^1"] \arrow[u, phantom, "\dashv"] & \mathbb{G}lob \arrow[u, shift left, "F^0"] \arrow[u, phantom, "\dashv"]
\arrow[from=1-2, to=2-2, shift left, "U^n"]
\arrow[from=1-3, to=2-3, shift left, "U^{n-1}"]
\arrow[from=1-4, to=2-4, shift left, "U^1"]
\arrow[from=1-5, to=2-5, shift left, "U^0"]
\end{tikzcd}$$

Thus, if we apply the functor U to this diagram in $\mathbb{A}dj$, we obtain the globular object in $\mathbb{M}nd$

$$\begin{tikzcd}
\cdots \arrow[r, dotted] & (\mathbb{G}lob^2, \omega^n) \arrow[r, shift left, "\sigma^n_{n-1}"] \arrow[r, shift right, "\beta^n_{n-1}"'] & (\mathbb{G}lob^2, \omega^{n-1}) \arrow[r, dotted] & (\mathbb{G}lob^2, \omega^1) \arrow[r, shift left, "\sigma^1_0"] \arrow[r, shift right, "\beta^1_0"'] & (\mathbb{G}lob, \omega^0)
\end{tikzcd}$$

and its underlying coglobular object $(\omega^\bullet, \delta, \kappa)$ of monads for the strict higher transformations

$$\begin{tikzcd}
\omega^0 \arrow[r, shift left, "\delta^1_0"] \arrow[r, shift right, "\kappa^1_0"'] & \omega^1 \arrow[r, shift left, "\delta^1_2"] \arrow[r, shift right, "\kappa^2_1"'] & \omega^2 \arrow[r, dotted] & \omega^{n-1} \arrow[r, shift left, "\delta^n_{n-1}"] \arrow[r, shift right, "\kappa^n_{n-1}"'] & \omega^n \arrow[r, dotted] & \cdots
\end{tikzcd}$$

An important coglobular of the strict higher transformations is given by the diagram

$$\begin{tikzcd}
\omega^0(1) \arrow[r, shift left, "\delta^1_0"] \arrow[r, shift right, "\kappa^1_0"'] & \omega^1(1,1) \arrow[r, shift left, "\delta^1_2"] \arrow[r, shift right, "\kappa^2_1"'] & \omega^2(1,1) \arrow[r, dotted] & \omega^{n-1}(1,1) \arrow[r, shift left, "\delta^n_{n-1}"] \arrow[r, shift right, "\kappa^n_{n-1}"'] & \omega^n(1,1) \arrow[r, dotted] & \cdots
\end{tikzcd}$$

where $(1,1)$ is the terminal object of the category $\mathbb{G}lob^2$. We abusively denote it by $(\omega^\bullet(1,1), \delta, \kappa)$. For each integer $n \geq 1$,

$$\omega^n(1,1) = (\omega^n_0(1,1), \omega^n_1(1,1))$$

is the free strict n-transformation on the terminal object $(1,1)$ of the category $\mathbb{G}lob^2$. These free strict n-transformations $\omega^n(1,1)$ $(n \geq 1)$ have an underlying $(n+1)$-globular set of maps which is denoted by

$$\begin{tikzcd}
\{\xi_n\} \arrow[r, shift left, "\sigma^{n+1}_n"] \arrow[r, shift right, "\beta^{n+1}_n"'] & \{\alpha_{n-1}, \beta_{n-1}\} \arrow[r, shift left, "\sigma^n_{n-1}"] \arrow[r, shift right, "\beta^n_{n-1}"'] & \cdots\cdots \arrow[r, shift left, "\sigma^{k+2}_{k+1}"] \arrow[r, shift right, "\beta^{k+2}_{k+1}"'] & \{\alpha_k, \beta_k\} \arrow[r, shift left, "\sigma^{k+1}_k"] \arrow[r, shift right, "\beta^{k+1}_k"'] & \cdots\cdots
\end{tikzcd}$$

$$\begin{tikzcd}
\cdots\cdots\cdots \arrow[r, shift left, "\sigma^3_2"] \arrow[r, shift right, "\sigma^3_2"'] & \{\alpha_1, \beta_1\} \arrow[r, shift left, "\sigma^2_1"] \arrow[r, shift right, "\sigma^2_1"'] & \{f, g\} \arrow[r, shift left, "\sigma^1_0"] \arrow[r, shift right, "\beta^1_0"'] & \{C, D\}.
\end{tikzcd}$$

where $C(0)\xrightarrow{\xi_n}D(n)$, for each integer $k\in[\![1,n-1]\!]$ we have the maps $C(0)\overset{\alpha_k}{\underset{\beta_k}{\rightrightarrows}}D(k)$, $C\overset{f}{\underset{g}{\rightrightarrows}}D$ are underlying strict higher functors which are its source and target, C and D represent its underlying strict higher categories source and target. This description of these free strict n-transformations $\omega^n(1,1)$ $(n\in\mathbb{N})$ gives a quick understanding of the maps δ^n_{n-1} and κ^n_{n-1}, which respectively send the free $(n-1)$-transformation ξ_{n-1} to the $(n-1)$-transformations α_{n-1} and β_{n-1}.

More precisely, for each $n\geq 1$, this description of $\omega^n(1,1)$ is a tool to describe the coglobular object $(\omega^{\bullet}(1,1),\delta,\kappa)$ just above: For each integer $n\geq 2$ we have $\delta^{n+1}_n(\xi_n)=\sigma^{n+1}_n(\xi_{n+1})$, where σ^{n+1}_n is the functor defined in Proposition 2.5. With the description just above for $\omega^{n+1}(1,1)$ and $\omega^n(1,1)$, it gives $\delta^{n+1}_n(\xi_n)=\alpha_n$. Also $\kappa^{n+1}_n(\xi_n)=\beta^{n+1}_n(\xi_{n+1})=\beta_n$ where β^{n+1}_n is the functor defined in Proposition 2.5. Also δ^1_0 sends the free strict higher category $\omega^0(1)$ to the domain C of the strict higher functor $f=\omega^1(1,1)$, and κ^1_0 sends the free strict higher category $\omega^0(1)$ to the codomain D of the strict higher functor $f=\omega^1(1,1)$. Finally, δ^2_1 sends the free strict higher functor $f=\omega^1(1,1)$ to the domain f of the free strict higher natural transformation $\xi_1=\omega^2(1,1)$, and κ^2_1 sends the free strict higher functor $f=\omega^1(1,1)$ to the codomain g of the free strict higher natural transformation $\xi_1=\omega^2(1,1)$.

The universal unit η^n of the monad ω^n of the strict n-transformations gives the morphism $(1,1)\xrightarrow{\eta^n(1,1)}\omega^n(1,1)$ in $\mathbb{G}lob^2$ which is in fact given by two morphisms $1\xrightarrow{\eta^n_0(1,1)}\omega^n_0(1,1)$ and $1\xrightarrow{\eta^n_1(1,1)}\omega^n_1(1,1)$ in $\mathbb{G}lob$. For each integer $n\in\mathbb{N}$, the unique n-cell of the terminal globular set is denoted by $1(n)$, also we use the shorter notations $1(n)=\eta^n_0(1,1)(1(n))$ and $2(n)=\eta^n_1(1,1)(1(n))$.

Remark 2.6. We deliberately describe this free strict n-transformation $\omega^n(1,1)$ with almost the same notations for m-cells of the underlying globular sets of C^n for $2\leq m\leq n+1$, except for functors, where here we use f and g for the underlying strict higher functors of $\omega^n(1,1)$, whereas symbols of functors of C^n are denoted by F and G. Also the underlying strict higher category C of $\omega^n(1,1)$ is just $\omega^n_0(1,1)$, and D is the free strict higher subcategory of $\omega^n_1(1,1)$ generated by the image of the universal map $\eta^n_1(1,1)$, that is, by all m-cells $2(m)$ $(m\in\mathbb{N})$.

3 The pointed ω^n-collections

If S is a cartesian monad on a category $\mathcal{G}$ then S-*collections* are kind of S-graphs defined in [14], where their domains of arities is an object $S(1)$ such that 1 is a terminal object of the category $\mathcal{G}$. The category of S-collections is denoted S-$\mathbb{C}oll$. The category of pointed S-collections is also defined in [14] and is denoted by S-$\mathbb{C}oll_p$. In this section we work with the locally finitely presentable category (l.f.p category) ω^n-$\mathbb{C}oll_p$ of pointed ω^n-collections ($n \in \mathbb{N}$). If $n \geq 1$ an object of ω^n-$\mathbb{C}oll_p$ is denoted by $(C, a, c; p)$, and described by a commutative diagram in $\mathbb{G}lob^2$

$$\begin{tikzcd}
 & (1,1) \arrow[ld, "\eta^n(1,1)"'] \arrow[d, "p"] \arrow[rd, "id"] & \\
\omega^n(1,1) & C \arrow[l, "a"] \arrow[r, "c"'] & (1,1)
\end{tikzcd}$$

and if $n = 0$ described by a commutative diagram in $\mathbb{G}lob$

$$\begin{tikzcd}
 & 1 \arrow[ld, "\eta^0(1)"'] \arrow[d, "p"] \arrow[rd, "id"] & \\
\omega^0(1) & C \arrow[l, "a"] \arrow[r, "c"'] & 1
\end{tikzcd}$$

The categories ω^n-$\mathbb{C}oll_p$ are monoidal, and the monoids in them are ω^n-operads used in Section 4. For tensors of all monoidal categories ω^n-$\mathbb{C}oll_p$ ($n \in \mathbb{N}$) we refer to [14]. We shall also use the category $\omega^\bullet$-$\mathbb{C}oll_p$ whose objects are the pointed ω^n-collections for each $n \in \mathbb{N}$, and morphisms of $\omega^\bullet$-$\mathbb{C}oll_p$ are those of ω^n-$\mathbb{C}oll_p$ for each $n \in \mathbb{N}$ plus morphisms

$$(C, a, c; p) \xrightarrow{(f,h)} (C', a', c'; p')$$

between the pointed ω^n-collections and the pointed ω^m-collections, where the underlying maps

$$C \xrightarrow{f} C' \text{ and } \omega^n(1,1) \xrightarrow{h} \omega^m(1,1)$$

make commutative diagrams. Also if $n = 0$ or $m = 0$, we have morphisms $(C, a, c; p) \xrightarrow{(f,h,q)} (C', a', c'; p')$ with extra maps $1 \xrightarrow{q} (1,1)$

(if $n = 0$) or $(1,1) \xrightarrow{q} 1$ (if $m = 0$) which also make commutative the corresponding diagrams.

In [7, 10] we described a coglobular object

$$\begin{tikzcd}
C^0 \arrow[r, shift left, "\delta_0^1"] \arrow[r, shift right, "\kappa_0^1"'] & C^1 \arrow[r, shift left, "\delta_2^1"] \arrow[r, shift right, "\kappa_1^2"'] & C^2 \arrow[r, dotted] & C^{n-1} \arrow[r, shift left, "\delta_{n-1}^n"] \arrow[r, shift right, "\kappa_{n-1}^n"'] & C^n \arrow[r, dotted] & {}
\end{tikzcd}$$

in the category of the pointed ω^0-collections, where for each $n \in \mathbb{N}$, C^n is a globular set which contains all symbols of operations needed for n-transformations. Actually, for each $n \geq 1$, each globular set C^n is isomorphic in $\mathbb{G}lob$ to the sum $C_0^n \sqcup C_1^n$ where C_0^n is the subglobular set of C^n containing all the symbols "μ" of operations needed for domain higher categories[9] of higher functors, plus the unary operation symbols "u", and C_1^n is the subglobular set of C^n containing all other symbols of operations needed for n-transformations, that is, all the symbols "ν" of operations of codomain higher categories of higher functors, plus all symbols "F" and "G" of operations for domain and codomain of higher natural transformations, plus other symbols "α", "β" and "ξ_n" of operations specific to n-transformations, and finally it contains the unary operation symbols "v". The unary operation symbols "u" and "v" give the pointing of C^n when it is seen as a collection. In the article [7][10], C^n are seen as pointed collection over the categorical sum $\omega^0(1) \sqcup \omega^0(1)$ for $n \geq 1$, and C^0 is the composition system of Batanin.

In this article, instead of using objects $C^n \simeq C_0^n \sqcup C_1^n$ in $\mathbb{G}lob$, we use objects in $\mathbb{G}lob^2$ that we still denote C^n, and which are defined by $C^n = (C_0^n, C_1^n)$, where the globular sets C_0^n and C_1^n are exactly those just above that we used in [7] to describe the "old" C^n.

Also another slightly but important difference with the C^n described in [7, 10], is that C^n here are seen as pointed ω^n-collections for each $n \in \mathbb{N}$ (see 3) instead of only being pointed ω^0-collections.

Thus the combinatorics is entirely similar to those of [7, 10], but for the convenience of the reader we are going to recall its precise constructions. Pointings are denoted by $(1,1) \xrightarrow{p^n} C^n$ if $n \geq 1$, and $1 \xrightarrow{p^0} C^0$ if $n = 0$.

---

[9] In fact, it is just the composition system defined in [2].

[10] In [7] $(C^n, a^n, c^n; p^n)$ denotes the pointed collection with underlying globular set C^n of operations symbols, but we can denote it just C^n when there is no risk of confusion. In the present article $C^n = (C_0^n, C_1^n)$ is seen as an object of $\mathbb{G}lob^2$ or is seen as a pointed ω^n-collection (see 3).

C^0 is a globular set and contains the symbols $\mu_p^m \in C^0(m)(0 \leq p < m)$ for the compositions of higher categories, plus the operadic unary symbols $u_m \in C^0(m)$. More specifically:

$\forall m \in \mathbb{N}$, C^0 contains an m-cell u_m such that: $s_{m-1}^m(u_m) = t_{m-1}^m(u_m) = u_{m-1}$ (if $m \geq 1$).

$\forall m \in \mathbb{N} - \{0, 1\}$, $\forall p \in \mathbb{N}$, such that $m > p$, C^0 contains an m-cell μ_p^m such that: If $p = m - 1$, $s_{m-1}^m(\mu_{m-1}^m) = t_{m-1}^m(\mu_{m-1}^m) = u_{m-1}$. If $0 \leq p < m - 1$, $s_{m-1}^m(\mu_p^m) = t_{m-1}^m(\mu_p^m) = \mu_p^{m-1}$.

Furthermore C^0 contains a 1-cell μ_0^1 such that $s_0^1(\mu_0^1) = t_0^1(\mu_0^1) = u_0$.

Pointing p^0 of C^0 is given by $p^0(1(m)) = u_m$ for all $m \in \mathbb{N}$.

For all integer $n \geq 1$, $C^n = (C_0^n, C_1^n)$ is such that $C_0^n = C^0$, thus we just need to describe the second component C_1^n of C^n. All such globular set C_1^n contains the following cells:

$\forall m \in \mathbb{N}$, C_1^n contains an m-cell v_m such that: $s_{m-1}^m(v_m) = t_{m-1}^m(v_m) = v_{m-1}$ (if $m \geq 1$).

$\forall m \in \mathbb{N} - \{0, 1\}$, $\forall p \in \mathbb{N}$, such that $m > p$, C_1^n contains an m-cell ν_p^m such that: If $p = m - 1$, $s_{m-1}^m(\nu_{m-1}^m) = t_{m-1}^m(\nu_{m-1}^m) = v_{m-1}$. If $0 \leq p < m - 1$, $s_{m-1}^m(\nu_p^m) = t_{m-1}^m(\nu_p^m) = \nu_p^{m-1}$.

Furthermore C_1^n contains a 1-cell ν_0^1 such that $s_0^1(\nu_0^1) = t_0^1(\nu_0^1) = v_0$,

plus other extra symbols of operations:

If $n = 1$, the globular set C_1^1 is built by adding to it a single symbol for functor (for each cell level): $\forall m \in \mathbb{N}$ the F^m m-cell is added, which is such that: If $m \geq 1$, $s_{m-1}^m(F^m) = t_{m-1}^m(F^m) = F^{m-1}$.

If $n = 2$, the globular set C_1^2 is built by adding to it two symbols of functor (for each cell level) and a symbol of natural transformation. More precisely

$\forall m \in \mathbb{N}$ we add the m-cell F^m such that: If $m \geq 1$, $s_{m-1}^m(F^m) = t_{m-1}^m(F^m) = F^{m-1}$, and we add the m-cell G^m such that: If $m \geq 1$, $s_{m-1}^m(G^m) = t_{m-1}^m(G^m) = G^{m-1}$.

And finally we add a 1-cell τ such that: $s_0^1(\tau) = F^0$ and $t_0^1(\tau) = G^0$.

If $n \geq 3$, each globular set C^n_1 is built by adding to it the required cells, specific to n-transformations. They contain four large groups of cells: the symbols of codomain higher categories, the symbols of the higher functors (domains and codomains), and the symbols of the n-transformations (natural higher transformations, higher modifications, etc). More precisely:

Symbols for higher functors $\forall m \in \mathbb{N}$, C^n_1 contains m-cells F^m and G^m such that: if $m \geq 1$, then $s^m_{m-1}(F^m) = t^m_{m-1}(F^m) = F^{m-1}$ and $s^m_{m-1}(G^m) = t^m_{m-1}(G^m) = G^{m-1}$.

Symbols for higher n-transformations $\forall p$, with $1 \leq p \leq n-1$, C^n_1 contains p-cells α_p and β_p which are such that: $\forall p$ with $2 \leq p \leq n-1$, $s^p_{p-1}(\alpha_p) = s^p_{p-1}(\beta_p) = \alpha_{p-1}$ and $t^p_{p-1}(\alpha_p) = t^p_{p-1}(\beta_p) = \beta_{p-1}$. If $p = 1$, then $s^1_0(\alpha_1) = s^1_0(\beta_1) = \alpha^0_0$ and $t^1_0(\alpha_1) = t^1_0(\beta_1) = \beta^0_0$. Finally C^n_1 contains an n-cell ξ_n such that $s^n_{n-1}(\xi_n) = \alpha_{n-1}$, $b^n_{n-1}(\xi_n) = \beta_{n-1}$.

For each $n \geq 1$, pointing of C^n is given by two morphisms $1 \xrightarrow{p^n_0} C^n_0$ and $1 \xrightarrow{p^n_1} C^n_1$ in $\mathbb{G}lob$ defined by: $p^n_0(1(m)) = u_m$ and $p^n_1(1(m)) = v_m$ for all $m \in \mathbb{N}$.

Let us denote by ω^0 the monad of the strict higher categories on the category $\mathbb{G}lob$, and let us denote by 1 the terminal globular set. Then the free strict higher category $\omega^0(1)$ was used by Batanin in [2] to define a pointed ω^0-collection

$$\begin{tikzcd}
 & 1 \arrow[dl, "\eta^0(1)"'] \arrow[d, "p^0"] \arrow[dr, "id"] & \\
\omega^0(1) & C^0 \arrow[l, "a^0"] \arrow[r, "c^0"'] & 1
\end{tikzcd}$$

called composition system in [2, 7]. Let us recall the definition of this pointed ω^0-collection: C^0 has been described above, and for all $0 \leq p < m$, $a^0(\mu^m_p) = 1(m) \star^m_p 1(m)$, and c^0 is just the unique map from C^0 to 1. This pointed ω^0-collection generates the operad B^0_C of Batanin whose algebras are his definition of weak higher categories. Also in [9] with the notion of strictly contractibility and contractible units, we can see that such ω^0-collection generates the monad ω^0, seen as operad over itself, that we denoted by $B^0_{S_u}$. In other word the terminal operad over ω^0 can be presented as the initial object in the category of operads over ω^0 which are strictly contractibles with contractible units and equipped with a C^0-system (see 4.3).

Let $(1,1)$ be the terminal object of the category $\mathbb{G}lob^2$. For each $n \geq 1$ we are going to describe a specific pointed ω^n-collection

$$\begin{tikzcd}
 & (1,1) \arrow[ld, "\eta^n(1,1)"'] \arrow[d, "p^n"] \arrow[rd, "id"] & \\
\omega^n(1,1) & C^n \arrow[l, "a^n"] \arrow[r, "c^n"'] & (1,1)
\end{tikzcd}$$

which generates the operads B^n_C of weak n-transformations. For $n = 1$ it generates the operads B^1_C of weak higher functors. Similarly to $B^0_{S_u}$, these ω^n-collections present operads ω^n of strict n-transformations (for each $n \geq 1$) as initial object $B^n_{S_u}$ of the category of operads over ω^n which are strictly contractibles, with contractibles units, and equipped with a C^n-system (see 5.2).

For each $n \geq 1$, let us denote by $(C^n, a^n, c^n; p^n)$, or C^n for short, such pointed ω^n-collection. We use the notation in the end of Section 2.2 to describe these pointed ω^n-collections $(C^n, a^n, c^n; p^n)$ $(n \geq 1)$:

If $n = 1$, the pointed ω^1-collection $(C^1, a^1, c^1; p^1)$ is given by two morphisms of spans in $\mathbb{G}lob$

$$\begin{tikzcd}
 & 1 \arrow[ld, "\eta^1_0(1,1)"'] \arrow[d, "p^1_0"] \arrow[rd, "id"] & \\
\omega^1_0(1,1) & C^1_0 \arrow[l, "a^1_0"] \arrow[r, "c^1_0"'] & 1
\end{tikzcd}
\qquad
\begin{tikzcd}
 & 1 \arrow[ld, "\eta^1_1(1,1)"'] \arrow[d, "p^1_1"] \arrow[rd, "id"] & \\
\omega^1_1(1,1) & C^1_1 \arrow[l, "a^1_1"] \arrow[r, "c^1_1"'] & 1
\end{tikzcd}$$

For all $m \in \mathbb{N}$, $a^1_0(u_m) = 1(m)$ and $c^1_0(u_m) = 1(m)$. For all integers $0 \leq p < m$, $a^1_0(\mu^m_p) = 1(m) \star^m_p 1(m)$ and $c^1_0(\mu^m_p) = 1(m)$. Similarly, for all $m \in \mathbb{N}$, $a^1_1(v_m) = 2(m)$ and $c^1_1(v_m) = 2(m)$. For all integers $0 \leq p < m$, $a^1_1(\nu^m_p) = 2(m) \star^m_p 2(m)$ and $c^1_1(\nu^m_p) = 2(m)$. Finally for all $m \in \mathbb{N}$, $a^1_1(F^m) = f^m(1(m))$ and $c^1_1(F^m) = 2(m)$.

If $n = 2$, the pointed ω^2-collection $(C^2, a^2, c^2; p^2)$ is given by two morphisms of spans in $\mathbb{G}lob$

$$\begin{tikzcd}
& 1 \arrow[dl, "\eta_0^2(1,1)"'] \arrow[d, "p_0^2"] \arrow[dr, "id"] & \\
\omega_0^2(1,1) & C_0^2 \arrow[l, "a_0^2"] \arrow[r, "c_0^2"'] & 1
\end{tikzcd}
\qquad
\begin{tikzcd}
& 1 \arrow[dl, "\eta_1^2(1,1)"'] \arrow[d, "p_1^2"] \arrow[dr, "id"] & \\
\omega_1^2(1,1) & C_1^2 \arrow[l, "a_1^2"] \arrow[r, "c_1^2"'] & 1
\end{tikzcd}$$

such that: For all $m \in \mathbb{N}$, $a_0^2(u_m) = 1(m)$ and $c_0^2(u_m) = 1(m)$. For all integers $0 \leq p < m$, $a_0^2(\mu_p^m) = 1(m) \star_p^m 1(m)$ and $c_0^2(\mu_p^m) = 1(m)$. Similarly for all $m \in \mathbb{N}$, $a_1^2(v_m) = 2(m)$ and $c_1^2(v_m) = 2(m)$. For all integers $0 \leq p < m$, $a_1^2(\nu_p^m) = 2(m) \star_p^m 2(m)$ and $c_1^2(\nu_p^m) = 2(m)$. Furthemore for all $m \in \mathbb{N}$, $a_1^2(F^m) = f^m(1(m))$ and $c_1^2(F^m) = 2(m)$, also $a_1^2(G^m) = g^m(1(m))$ and $c_1^2(G^m) = 2(m)$. Finally we have $a_1^2(\tau) = \tau(1(0))$ and $c_1^2(\tau) = 2(1)$.

If $n \geq 3$, the pointed ω^n-collection $(C^n, a^n, c^n; p^n)$ is given by two morphisms of spans in $\mathbb{G}lob$

$$\begin{tikzcd}
& 1 \arrow[dl, "\eta_0^n(1,1)"'] \arrow[d, "p_0^n"] \arrow[dr, "id"] & \\
\omega_0^n(1,1) & C_0^n \arrow[l, "a_0^n"] \arrow[r, "c_0^n"'] & 1
\end{tikzcd}
\qquad
\begin{tikzcd}
& 1 \arrow[dl, "\eta_1^n(1,1)"'] \arrow[d, "p_1^n"] \arrow[dr, "id"] & \\
\omega_1^n(1,1) & C_1^n \arrow[l, "a_1^n"] \arrow[r, "c_1^n"'] & 1
\end{tikzcd}$$

such that: For all $m \in \mathbb{N}$, $a_0^n(u_m) = 1(m)$ and $c_0^n(u_m) = 1(m)$. For all integers $0 \leq p < m$, $a_0^n(\mu_p^m) = 1(m) \star_p^m 1(m)$ and $c_0^n(\mu_p^m) = 1(m)$. Similarly for all $m \in \mathbb{N}$, $a_1^n(v_m) = 2(m)$ and $c_1^n(v_m) = 2(m)$. For all integers $0 \leq p < m$, $a_1^n(\nu_p^m) = 2(m) \star_p^m 2(m)$ and $c_1^n(\nu_p^m) = 2(m)$. Furthemore for all $m \in \mathbb{N}$, $a_1^n(F^m) = f^m(1(m))$ and $c_1^n(F^m) = 2(m)$, and also $a_1^n(G^m) = g^m(1(m))$ and $c_1^n(G^m) = 2(m)$. Furthemore, for all $1 \leq k \leq n-1$: $a_1^n(\alpha_k) = \alpha_k(1(0))$, and $c_1^n(\alpha_k) = 2(k)$, and also $a_1^n(\beta_k) = \beta_k(1(0))$, and $c_1^n(\beta_k) = 2(k)$. Finally we have $a_1^n(\xi_n) = \xi_n(1(0))$ and $c_1^n(\xi_n) = 2(n)$.

Now let us describe the coglobular object $C^\bullet$ in $\omega^\bullet$-$\mathbb{C}oll_p$, which is very similar to the one described in [7, 9, 11]

$$\begin{tikzcd}
C^0 \arrow[r, shift left, "\delta^1_0"] \arrow[r, shift right, "\kappa^1_0"'] \arrow[d, "a^0"'] & C^1 \arrow[r, shift left, "\delta^1_2"] \arrow[r, shift right, "\kappa^2_1"'] \arrow[d, "a^1"'] & C^2 \arrow[r, dotted] \arrow[d, "a^2"'] & C^{n-1} \arrow[r, shift left, "\delta^n_{n-1}"] \arrow[r, shift right, "\kappa^n_{n-1}"'] \arrow[d, "a^{n-1}"'] & C^n \arrow[r, dotted] \arrow[d, "a^n"'] & {} \\
\omega^0(1) \arrow[r, shift left, "\delta^1_0"] \arrow[r, shift right, "\kappa^1_0"'] & \omega^1(1,1) \arrow[r, shift left, "\delta^1_2"] \arrow[r, shift right, "\kappa^2_1"'] & \omega^2(1,1) \arrow[r, dotted] & \omega^{n-1}(1,1) \arrow[r, shift left, "\delta^n_{n-1}"] \arrow[r, shift right, "\kappa^n_{n-1}"'] & \omega^n(1,1) \arrow[r, dotted] & {}
\end{tikzcd}$$

For all $k \in \mathbb{N}$, we deliberately denote by the same notation morphisms $C^k \underset{\kappa_k^{k+1}}{\overset{\delta_k^{k+1}}{\rightrightarrows}} C^{k+1}$ and morphisms $\omega^k(1,1) \underset{\kappa_k^{k+1}}{\overset{\delta_k^{k+1}}{\rightrightarrows}} \omega^{k+1}(1,1)$. In fact it is not difficult to check (see below) that this coglobular object $(C^\bullet, \delta, \kappa)$ is build in a way that, for all $k \in \mathbb{N}$, $\delta_k^{k+1} \circ a^k = a^{k+1} \circ \delta_k^{k+1}$, $\kappa_k^{k+1} \circ a^k = a^{k+1} \circ \kappa_k^{k+1}$, but also for all integer $k \in \mathbb{N}$, we have the equalities $c^{k+1} \circ \delta_k^{k+1} = c^{k+1} \circ \kappa_k^{k+1} = c^k$.

The morphism $C^0 \xrightarrow{\delta_0^1} C^1$ is given by the morphism $C^0 \xrightarrow{\delta_0^1} C_0^1$ in $\mathbb{G}lob$ which sends, for all $m \in \mathbb{N}$, the m-cell u_m of C^0 to the m-cell u_m of C_0^1. Also it sends, for all integers $0 \leq p < m$, all m-cells μ_p^m of C^0 to the m-cells μ_p^m of C_0^1. The morphism $C^0 \xrightarrow{\kappa_0^1} C^1$ is given by the morphism $C^0 \xrightarrow{\kappa_0^1} C_1^1$ in $\mathbb{G}lob$, which sends, for all $m \in \mathbb{N}$, all m-cells u_m of C^0 to the m-cells v_m of C_1^1. Also it sends, for all integers $0 \leq p < m$, all m-cells μ_p^m of C^0 to the m-cells ν_p^m of C_1^1.

The morphism $C^1 \xrightarrow{\delta_1^2} C^2$ is given by two morphisms $C_0^1 \xrightarrow{\delta_{1,0}^2} C_0^2$ and $C_1^1 \xrightarrow{\delta_{1,1}^2} C_1^2$ in $\mathbb{G}lob$ such that: $\delta_{1,0}^2$ sends, for all $m \in \mathbb{N}$, the m-cell u_m of C_0^1 to the m-cell u_m of C_0^2, and $\delta_{1,1}^2$ sends, for all $m \in \mathbb{N}$, the m-cell v_m of C_1^1 to the m-cell v_m of C_1^2. Also $\delta_{1,0}^2$ sends, for all integers $0 \leq p < m$, the m-cell μ_p^m of C_0^1 to the m-cell μ_p^m of C_0^2, and $\delta_{1,1}^2$ sends, for all integers $0 \leq p < m$, the m-cell ν_p^m of C_1^1 to the m-cell ν_p^m of C_1^2. Furthemore $\delta_{1,1}^2$ sends, for all $m \in \mathbb{N}$, the m-cell F^m of C_1^1 to the m-cell F^m of C_1^2.

The morphism $C^1 \xrightarrow{\kappa_1^2} C^2$ is given by two morphisms $C_0^1 \xrightarrow{\kappa_{1,0}^2} C_0^2$ and $C_1^1 \xrightarrow{\kappa_{1,1}^2} C_1^2$ in $\mathbb{G}lob$ such that: $\kappa_{1,0}^2$ sends, for all $m \in \mathbb{N}$, the m-cell

u_m of C^1_0 to the m-cell u_m of C^2_0, and $\kappa^2_{1,1}$ sends, for all $m \in \mathbb{N}$, the m-cell v_m of C^1_1 to the m-cell v_m of C^2_1. Also $\kappa^2_{1,0}$ sends, for all integers $0 \leq p < m$, the m-cell μ^m_p of C^1_0 to the m-cell μ^m_p of C^2_0, and $\kappa^2_{1,1}$ sends, for all integers $0 \leq p < m$, the m-cell ν^m_p of C^1_1 to the m-cell ν^m_p of C^2_1. Furthemore $\kappa^2_{1,1}$ sends, for all $m \in \mathbb{N}$, the m-cell F^m of C^1_1 to the m-cell G^m of C^2_1.

For all $n \geq 2$, we define the morphisms $C^n \xrightarrow{\delta^{n+1}_n} C^{n+1}$ and $C^n \xrightarrow{\kappa^{n+1}_n} C^{n+1}$.

The morphism $C^n \xrightarrow{\delta^{n+1}_n} C^{n+1}$ is given by two morphisms $C^n_0 \xrightarrow{\delta^{n+1}_{n,0}} C^{n+1}_0$ and $C^n_0 \xrightarrow{\delta^{n+1}_{n,1}} C^{n+1}_1$ in $\mathbb{G}lob$ such that: $\delta^{n+1}_{n,0}$ sends, for all $m \in \mathbb{N}$, the m-cell u_m of C^n_0 to the m-cell u_m of C^{n+1}_0, for all integers $0 \leq p < m$, the m-cell μ^m_p of C^n_0 to the m-cell μ^m_p of C^{n+1}_0. $\delta^{n+1}_{n,1}$ sends, for all $m \in \mathbb{N}$, the m-cell v_m of C^n_1 to the m-cell v_m of C^{n+1}_1, for all integers $0 \leq p < m$, the m-cell ν^m_p of C^n_1 to the m-cell ν^m_p of C^{n+1}_1. For all $1 \leq k < n$, $\delta^{n+1}_{n,1}$ sends the k-cells α_k and β_k of C^n_1 respectively to the k-cells α_k and β_k of C^{n+1}_1. Furthemore $\delta^{n+1}_{n,1}$ sends the n-cell ξ_n of C^n_1 to the n-cell α_n of C^{n+1}_1.

The morphism $C^n \xrightarrow{\kappa^{n+1}_n} C^{n+1}$ is given by two morphisms $C^n_0 \xrightarrow{\kappa^{n+1}_{n,0}} C^{n+1}_0$ and $C^n_0 \xrightarrow{\kappa^{n+1}_{n,1}} C^{n+1}_1$ in $\mathbb{G}lob$ such that: $\kappa^{n+1}_{n,0}$ sends, for all $m \in \mathbb{N}$, the m-cell u_m of C^n_0 to the m-cell u_m of C^{n+1}_0, for all integers $0 \leq p < m$, the m-cell μ^m_p of C^n_0 to the m-cell μ^m_p of C^{n+1}_0. $\kappa^{n+1}_{n,1}$ sends, for all $m \in \mathbb{N}$, the m-cell v_m of C^n_1 to the m-cell v_m of C^{n+1}_1, for all integers $0 \leq p < m$, the m-cell ν^m_p of C^n_1 to the m-cell ν^m_p of C^{n+1}_1. For all $1 \leq k < n$, $\kappa^{n+1}_{n,1}$ sends the k-cells α_k and β_k of C^n_1 respectively to the k-cells α_k and β_k of C^{n+1}_1. Furthemore $\delta^{n+1}_{n,1}$ sends the n-cell ξ_n of C^n_1 to the n-cell β_n of C^{n+1}_1.

4 Higher operads of the weak n-transformations

If S is a cartesian monad on a category $\mathcal{G}$ then S-*operads* are kind of S-categories defined in [2, 5, 14] [11], where their domains of arities is an object $S(1)$ such that 1 is the terminal object of the category $\mathcal{G}$. The category of S-operads is denoted by S-$\mathbb{O}per$. In this section we work with the lo-

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[11] Claudio Hermida calls them *multicategories* in [5].

cally finitely presentable category ω^n-$\mathbb{O}per$ of ω^n-operads ($n \in \mathbb{N}$). An ω^n-operad is a monoid in the monoidal category ω^n-$\mathbb{C}oll_p$ (see Section 3) and is denoted by (B, a, c). A basic example of ω^n-operad is given by the terminal ω^n-operad ω^n of strict n-transformations (see Theorem 2.4). We shall use also the category $\omega^\bullet$-$\mathbb{O}per$ whose objects are ω^n-operads for each $n \in \mathbb{N}$, and morphisms of $\omega^\bullet$-$\mathbb{O}per$ are those of ω^n-$\mathbb{O}per$ for each $n \in \mathbb{N}$, plus morphisms $(B, a, c) \xrightarrow{(f,h)} (B', a', c')$ between ω^n-operads and ω^m-operads ($n, m \geq 1$), where the underlying maps $B \xrightarrow{f} B'$ and $\omega^n(1,1) \xrightarrow{h} \omega^m(1,1)$, make commutative diagrams and preserve monoid structures. Also if $n = 0$ or $m = 0$, we have morphisms

$$(B, a, c) \xrightarrow{(f,h,q)} (B', a', c')$$

with extra maps $1 \xrightarrow{q} (1,1)$ (if $n = 0$) or $(1,1) \xrightarrow{q} 1$ (if $m = 0$) which also make commutative the corresponding diagrams, and preserve monoid structures.

In [14] it is proved that for each $n \in \mathbb{N}$ the forgetful functor

$$\omega^n\text{-}\mathbb{O}per \xrightarrow{U^n} \omega^n\text{-}\mathbb{C}oll_p$$

is monadic. Consider $(C = (C_0, C_1), a = (a_0, a_1), c = (c_0, c_1))$ a fixed ω^n-collection[12]. For each integers $k \geqslant 1$, let us note $\tilde{C_0}(k) = \{(x, y) \in C_0(k) \times C_0(k) : x \| y \text{ and } a_0(x) = a_0(y)\}$ and $\tilde{C_1}(k) = \{(x, y) \in C_1(k) \times C_1(k) : x \| y \text{ and } a_1(x) = a_1(y)\}$. Also we put $\tilde{C_0}(0) = \{(x, y) \in C_0(0) \times C_0(0) : a_0(x) = a_0(y)\}$ and $\tilde{C_1}(0) = \{(x, y) \in C_1(0) \times C_1(0) : a_1(x) = a_1(y)\}$.

Definition 4.1. For each integer $n \in \mathbb{N}$, a contraction on the ω^n-collection (C, d, c), is the datum, for each $i \in \{0, 1\}$ and for all $k \in \mathbb{N}$, of a map $\tilde{C_i}(k) \xrightarrow{[,]_k} C_i(k+1)$ such that

- $s([\alpha, \beta]_k) = \alpha, t([\alpha, \beta]_k) = \beta$,
- $a_i([\alpha, \beta]_k) = 1^k_{k+1}(a_i(\alpha) = a_i(\beta))$.

---

[12] We do not need it to be pointed here.

An ω^n-collection which is equipped with a contraction will be called contractible and we use the notation $(C,a,c;([,]_k)_{k\in\mathbb{N}})$ for a contractible ω^n-collection.

A pointed contractible ω^n-collections is denoted by $(C,a,c;p,([,]_k)_{k\in\mathbb{N}})$ (compare with Section 1.2 of the article [7]), and morphisms between two pointed contractible ω^n-collections preserve contractibilities and pointings. The locally finitely presentable category of pointed contractible ω^n-collections is denoted by $C\omega^n\text{-}\mathbb{C}oll_p$, and the forgetful functor V^n

$$\begin{tikzcd} C\omega^n\text{-}\mathbb{C}oll_p \arrow[r, "V^n"] & \omega^n\text{-}\mathbb{C}oll_p \end{tikzcd}$$

is monadic. The functor H^n denotes its left adjoint.

An ω^n-operad is contractible if its underlying pointed ω^n-collection lies in $C\omega^n\text{-}\mathbb{C}oll_p$. Morphisms between two contractible ω^n-operads are morphisms of ω^n-operads which preserve contractibilities. Let us write $C\omega^n\text{-}\mathbb{O}per$ for the category of contractible ω^n-operads. Also consider the following pullback in $\mathbb{C}AT$:

$$\begin{tikzcd} C\omega^n\text{-}\mathbb{C}oll_p \underset{\omega^n\text{-}\mathbb{C}oll_p}{\times} \omega^n\text{-}\mathbb{O}per \arrow[r, "p_1"] \arrow[d, "p_2"'] & \omega^n\text{-}\mathbb{O}per \arrow[d, "U^n"] \\ C\omega^n\text{-}\mathbb{C}oll_p \arrow[r, "V^n"'] & \omega^n\text{-}\mathbb{C}oll_p \end{tikzcd}$$

If we apply the proposition of Max Kelly of Section 3 of [11] to the diagram above, it shows that we have an equivalence of categories

$$C\omega^n\text{-}\mathbb{C}oll_p \underset{\omega^n\text{-}\mathbb{C}oll_p}{\times} \omega^n\text{-}\mathbb{O}per \simeq C\omega^n\text{-}\mathbb{O}per$$

such that $C\omega^n\text{-}\mathbb{O}per$ is a locally presentable category, and also that the forgetful functor

$$\begin{tikzcd} C\omega^n\text{-}\mathbb{O}per \arrow[r, "O^n"] & \omega^n\text{-}\mathbb{C}oll_p \end{tikzcd}$$

is monadic. Denote by F^n the left adjoint of O^n.

Remark 4.2. The morphisms of the category $C\omega^n\text{-}\mathbb{O}per$ preserve contractions.

Units of each ω^n-operad equipped it with a canonical pointing. This fact leads to the following definition:

Definition 4.3. An ω^n-operad B has a C^n-system if there exist a morphism $C^n \xrightarrow{s^n} B$ in the category ω^n-$\mathbb{C}oll_p$.

For all $n \in \mathbb{N}$ the category of contractible ω^n-operads equipped with a C^n-system is denoted by $CC^n\omega^n$-$\mathbb{O}per$. Its morphisms preserve contractions and their C^n-systems and it is a locally finitely presentable category.

Definition 4.4. If $n = 1$, the free contractible ω^1-operad

$$(B^1_C, a^1, c^1) = F^1(C^1, a^1, c^1; p^1)$$

on the pointed ω^1-collection $(C^1, a^1, c^1; p^1)$ is the intial object in the category $CC^1\omega^n$-$\mathbb{O}per$ of contractible ω^1-operads equipped with a C^1-system. The universal map η^1 of B^1_C gives the pointing of its underlying pointed ω^1-collection. It is the higher operad of the weak higher functors.

Definition 4.5. For all $n \geq 2$, the free contractible ω^n-operad

$$(B^n_C, a^n, c^n) = F^n(C^n, a^n, c^n; p^n)$$

on the pointed ω^n-collection $(C^n, a^n, c^n; p^n)$ is the intial object in the category $CC^n\omega^n$-$\mathbb{O}per$ of contractible ω^n-operads equipped with a C^n-system. The universal map η^n of B^n_C gives the pointing of its underlying pointed ω^n-collection. It is the higher operad of the weak n-transformations

(B^n_C, a^n, c^n) is given by two diagrams in $\mathbb{G}lob$

$$\begin{tikzcd}
 & 1 \arrow[dl, "\eta^n_0(1{,}1)"'] \arrow[d, "p^n_0"] \arrow[dr, "id"] & \\
\omega^n_0(1,1) & C^n_0 \arrow[l, "a^n_0"'] \arrow[r, "c^n_0"] \arrow[d, "\eta^n_0"] & 1 \\
 & B^n_{C,0} \arrow[ul, "a^n_0"] \arrow[ur, "c^n_0"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & 1 \arrow[dl, "\eta^n_1(1{,}1)"'] \arrow[d, "p^n_1"] \arrow[dr, "id"] & \\
\omega^n_1(1,1) & C^n_1 \arrow[l, "a^n_1"'] \arrow[r, "c^n_1"] \arrow[d, "\eta^n_1"] & 1 \\
 & B^n_{C,1} \arrow[ul, "a^n_1"] \arrow[ur, "c^n_1"'] &
\end{tikzcd}$$

The operadic multiplication[13] $B^n_C \otimes B^n_C \xrightarrow{\gamma^n} B^n_C$ is given by two morphisms in $\mathbb{G}lob$

$$\omega^n_0(B^n_C) \underset{\omega^n_0(1,1)}{\times} B^n_{C,0} \xrightarrow{\gamma^n_0} B^n_{C,0} \qquad \omega^n_1(B^n_C) \underset{\omega^n_1(1,1)}{\times} B^n_{C,1} \xrightarrow{\gamma^n_1} B^n_{C,1}$$

and $(B^n_C = (B^n_{C,0}, B^n_{C,1}), \eta^n = (\eta^n_1, \eta^n_0), \mu^n = (\mu^n_0, \mu^n_1))$ denotes its corresponding monad.

Consider an object (G_0, G_1) of the category $\mathbb{G}lob^2$. A B^n_C-algebra on (G_0, G_1) is given by a morphism $B^n_C(G_0, G_1) \xrightarrow{v} (G_0, G_1)$ in $\mathbb{G}lob^2$ which satisfies the usual axioms of algebras, and in particular it is given by two morphisms in the category $\mathbb{G}lob$

$$B^n_{C,0} \underset{\omega^n_0(1,1)}{\times} \omega^n_0(G_0, G_1) \xrightarrow{v_0} G_0 \qquad B^n_{C,1} \underset{\omega^n_1(1,1)}{\times} \omega^n_1(G_0, G_1) \xrightarrow{v_1} G_1$$

In Section 5 of the article [7] we defined a notion of dimension for B^n_C-algebras. For our context it goes as follow: A reflexive globular set H has dimension $p \in \mathbb{N}$ if all its q-cells for $q > p$ are identities, that is, are reflexivity of lower cells, and if there exist a least one p-cell in H which is not an identity. In that case we write $dim(H) = p$. Each B^n_C-algebra (G, v) equipped globular sets G_0 and G_1 with a reflexive structure defined as follows: Take $x \in G_0(n)$ and $y \in G_1(n)$, then $1^n_{n+1}(x) := v_0([u_n; u_n]; 1^n_{n+1}(\eta^n_0(x)))$ and $1^n_{n+1}(y) := v_1([u_n; u_n]; 1^n_{n+1}(\eta^n_1(y)))$, where η^n is the universal map for the monad ω^n (see 2).

Definition 4.6. For each $n \geq 1$, a B^n_C-algebra (G, v) has dimension p if its underlying reflexive globular sets G_0 and G_1 are such that

$$sup(dim(G_0), dim(G_1) = p.$$

In that case we write $dim(G, v) = p$.

As an exercise we encourage the reader to prove the following propositions whose proofs are similar to those we can find in [6, 7]:

Proposition 4.7. *Each B^1_C-algebra of dimension 2 is a pseudo-2-functor.*

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[13] Sometimes called *operadic composition.*

Proposition 4.8. *Each B^2_C-algebra of dimension* 2 *is a pseudo-*2*-natural transformation.*

Proposition 4.9. *Each B^3_C-algebra of dimension* 2 *is a pseudo-*2*-modification.*

5 Standard actions and hypotheses

Let us recall that, thanks to Proposition 7.2 of the article [2], if a coglobular higher operadic object $(W^\bullet, \delta, \kappa)$ lives in a category $\mathcal{C}$ with small pushouts then we can associate to it a coendomorphism operad $Coend(W)$ over the cartesian monad ω^0 of the strict ∞-categories. This proposition of Michael Batanin is the key technical point to replace the following hypothesis.

Conjecture 5.1. The weak higher category of the weak higher categories exists in the globular setting.

by one of the very precise technical Conjectures 5.5 and 5.6 described in Subsection 5.1 just below.

In this section we start by giving a slight generalization of technics described in Section 2 of the article [10]. Here $\mathbb{C}AT_{Push}$ denotes the category of categories having pushouts and with morphism functors which preserve pushouts, and $\mathcal{C}$ denotes a category of higher operads such that $\mathcal{C}$ is an object of $\mathbb{C}AT_{Push}$. Such a category $\mathcal{C}$ seen as a subcategory of $\mathbb{M}nd$ (see Section 2.2) must have small pullbacks, those which exist in $\mathbb{M}nd$. It is well known that the functor $\mathbb{M}nd \xrightarrow{(.)\text{-}\mathbb{A}lg} \mathbb{C}AT$ preserves limits, thus it preserves limits of $\mathcal{C}$ and thus it preserves pushouts of $\mathcal{C}$ when considering $\mathcal{C}$ only with its underlying class of monads and natural transformations between them, which is our context here.

We have the following diagram in $\mathbb{C}AT_{Push}$

$$\begin{tikzcd}\mathcal{C} \arrow[r, "(.)\text{-}\mathbb{A}lg"] & \mathbb{C}AT^{op} \arrow[r, "Ob(.)"] & SET^{op}.\end{tikzcd}$$

Given a coglobular higher operadic object $(W^\bullet, \delta, \kappa)$ in the category $\mathcal{C}$, it induces the following diagram in $\mathbb{C}AT_{Push}$

$$\begin{tikzcd}
 & \mathbb{G}_0 \arrow[ld, "W"'] \arrow[d, "A^{op}"] \arrow[rd, "A_0^{op}"] & \\
\mathcal{C} \arrow[r, "\mathbb{A}lg(.)"'] & \mathbb{C}AT^{op} \arrow[r, "Ob(.)"'] & SET^{op}.
\end{tikzcd}$$

If we apply the functor $Coend(.)$ of Proposition 1 and Corollary 2 of Section 1.4 of the article [10] to this diagram (just switch the category T-$\mathbb{C}\mathrm{at}_1$ with the category $\mathcal{C}$), we get the following definition.

Definition 5.2. The standard action associated to the coglobular object $(W^\bullet, \delta, \kappa)$ in $\mathcal{C}$ is defined by the following diagram of the category ω^0-$\mathbb{O}per$

$$\begin{tikzcd}
Coend(W) \arrow[r, "Coend((.)\text{-}\mathbb{A}lg)"] & Coend(A^{op}) \arrow[r, "Coend(Ob(.))"] & End(A_0)\ .
\end{tikzcd}$$

We also use the following definitions which generalize those in the article [10].

Definition 5.3. A coglobular higher operadic object $(W^\bullet, \delta, \kappa)$ in a category $\mathcal{C}$ with small pushouts is called *algebraic* if $W(0)$ is an object of ω^0-$\mathbb{O}per$ and if it is additionally equipped with a higher operadic morphism

$$\begin{tikzcd}
W(0) \arrow[r, "w"] & Coend(W)\ .
\end{tikzcd}$$

Definition 5.4. A higher operad A is *fractal* if there exists an algebraic coglobular higher operadic object of the form $(W^\bullet, \delta, \kappa)$ in a category $\mathcal{C}$ of higher operads with small pushouts, such that $W(0) = A$.

In [9–11] we gave relevant examples of algebraic coglobular object of higher operads and relevant examples of fractal higher operads.

5.1 The Violet Operad

Now lets us describe[14] the coglobular object $(B^\bullet_C, \delta, \kappa)$ in $\omega^\bullet$-$\mathbb{O}per$, which is very similar to the one described in [7, 9, 11]:

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[14] The category $\omega^\bullet$-$\mathbb{O}per$ must be seen only as a tool to describe properly the coglobular object $(B^\bullet_C, \delta, \kappa)$.

$$\begin{tikzcd}
B^0_C \arrow[r, shift left, "\delta^1_0"] \arrow[r, shift right, "\kappa^1_0"'] \arrow[d, "a^0"] & B^1_C \arrow[r, shift left, "\delta^1_2"] \arrow[r, shift right, "\kappa^2_1"'] \arrow[d, "a^1"] & B^2_C \arrow[r, dotted] \arrow[d, "a^2"] & B^{n-1}_C \arrow[r, shift left, "\delta^n_{n-1}"] \arrow[r, shift right, "\kappa^n_{n-1}"'] \arrow[d, "a^{n-1}"] & B^n_C \arrow[r, dotted] \arrow[d, "a^n"] & {} \\
\omega^0(1) \arrow[r, shift left, "\delta^1_0"] \arrow[r, shift right, "\kappa^1_0"'] & \omega^1(1,1) \arrow[r, shift left, "\delta^1_2"] \arrow[r, shift right, "\kappa^2_1"'] & \omega^2(1,1) \arrow[r, dotted] & \omega^{n-1}(1,1) \arrow[r, shift left, "\delta^n_{n-1}"] \arrow[r, shift right, "\kappa^n_{n-1}"'] & \omega^n(1,1) \arrow[r, dotted] & {}
\end{tikzcd}$$

For each integer $n \geq 1$, the morphism $B^n_C \xrightarrow{\delta^{n+1}_n} B^{n+1}_C$ is build as follows:

$$\begin{tikzcd}
C^n \arrow[rr, "\delta^{n+1}_n"] \arrow[d, "\eta^n"'] \arrow[dr, "!"] & & C^{n+1} \arrow[d, "\eta^{n+1}"] \\
B^n_C \arrow[r, "!'"] \arrow[d, "a^n"'] & B^{n+1}_C \underset{\omega^{n+1}(1,1)}{\times} \omega^n(1,1) \arrow[r, "p_2"] \arrow[dl, "p_1"] & B^{n+1}_C \arrow[d, "a^{n+1}"] \\
\omega^n(1,1) \arrow[rr, "\delta^{n+1}_n"'] & & \omega^{n+1}(1,1)
\end{tikzcd}$$

where the bottom left diagram is a pullback in $\omega^\bullet$-$\mathbb{C}oll$, and furthemore

$$B^{n+1}_C \underset{\omega^{n+1}(1,1)}{\times} \omega^n(1,1)$$

is still an ω^n-operad according to Section 6.7 of [14]. It is rather evident that it is still a contractible ω^n-operad, because the description of the bottom map δ^{n+1}_n shows that the couples $(b,x) \in B^{n+1}_C \underset{\omega^{n+1}(1,1)}{\times} \omega^n(1,1)$ are pairs such that $a^{n+1}(b) = x$, and two parallels cells $(b,x)||(b',x') \in B^{n+1}_C \underset{\omega^{n+1}(1,1)}{\times} \omega^n(1,1)$ with the same arities are such that $b||b'$ in B^{n+1}_C and which coherence $[b,b']$ in B^{n+1}_C has arity in the image[15] of δ^{n+1}_n, thus the coherence cell $[(b,x),(b',x')] \in B^{n+1}_C \underset{\omega^{n+1}(1,1)}{\times} \omega^n(1,1)$ is exactly the cell $(1_x,[b,b'])$. Because

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[15] The description of the coglobular object $(\omega^\bullet(1,1),\delta,\kappa)$ in 2.2 shows that δ^{n+1}_n sends the free strict n-transformation ξ_n to the free strict n-transformation α_n.

we have $\delta_n^{n+1} \circ a^n \circ \eta^n = a^{n+1} \circ \eta^{n+1} \circ \delta_n^{n+1}$, thus we have the existence of a unique map

$$C^n \xrightarrow{\quad ! \quad} B_C^{n+1} \underset{\omega^{n+1}(1,1)}{\times} \omega^n(1,1)$$

such that $p_1 \circ ! = a^n \circ \eta^n$ and $p_2 \circ ! = \eta^{n+1} \circ \delta^{n+1}$. It shows that $B_C^{n+1} \underset{\omega^{n+1}(1,1)}{\times} \omega^n(1,1)$ is a contractible ω^n-operad equipped with a C^n-system. Thus, the universality of the map $C^n \xrightarrow{\eta^n} B_C^n$ produces a unique map

$$B_C^n \xrightarrow{!'} B_C^{n+1} \underset{\omega^{n+1}(1,1)}{\times} \omega^n(1,1)$$

such that $!' \circ \eta^n =!$ and $p_1 \circ !' = a^n$. Thus, we obtain the morphism $B_C^n \xrightarrow{\delta_n^{n+1}} B_C^{n+1}$ defined by $\delta_n^{n+1} = p_2 \circ !'$. Similarly we can form the morphism $B_C^n \xrightarrow{\kappa_n^{n+1}} B_C^{n+1}$.

Consider the functor $\mathbb{M}nd \xrightarrow{A} \mathbb{A}dj$ of Section 2.2, and the functor $\mathbb{A}dj \xrightarrow{V} \mathbb{C}AT$ which sends an object $(F, G, \eta, \varepsilon)$ of $\mathbb{A}dj$ to A and a morphism $(F, G, \eta, \varepsilon) \xrightarrow{f} (F', G', \eta', \varepsilon')$ of $\mathbb{A}dj$ to the functor $A \xrightarrow{h} A'$ (see the notations of Section 2.2), then the functor $\mathbb{M}nd \xrightarrow{V \circ A} \mathbb{C}AT$ sends this coglobular object $(B_C^\bullet, \delta, \kappa)$ to the following globular object in $\mathbb{C}AT$:

$$\begin{tikzcd}
{} \arrow[r, dotted] & B_C^n\text{-}\mathbb{A}lg \arrow[r, shift left, "\sigma_{n-1}^n"] \arrow[r, shift right, "\beta_{n-1}^n"'] & B_C^{n-1}\text{-}\mathbb{A}lg \arrow[r, dotted] & B_C^1\text{-}\mathbb{A}lg \arrow[r, shift left, "\sigma_0^1"] \arrow[r, shift right, "\beta_0^1"'] & B_C^0\text{-}\mathbb{A}lg \, .
\end{tikzcd}$$

This is the globular category of the weak ∞-categories.

As in Section 2, we consider the functor $\mathbb{G}\mathbb{C}AT \xrightarrow{GLOB} \mathbb{G}LOB$ which sends a globular category to its object part. If we apply it to the globular category of the weak ∞-categories we obtain the following globular object in $\mathbb{S}ET$:

$$\begin{tikzcd}
{} \arrow[r, dotted] & B_C^n\text{-}\mathbb{A}lg(0) \arrow[r, shift left, "\sigma_{n-1}^n"] \arrow[r, shift right, "\beta_{n-1}^n"'] & B_C^{n-1}\text{-}\mathbb{A}lg(0) \arrow[r, dotted] & B_C^1\text{-}\mathbb{A}lg(0) \arrow[r, shift left, "\sigma_0^1"] \arrow[r, shift right, "\beta_0^1"'] & B_C^0\text{-}\mathbb{A}lg(0) \, .
\end{tikzcd}$$

As we said in the introduction, one of the most important conjecture in higher category theory is to prove that the weak ∞-category of the weak ∞-categories exists in the globular setting. We believe the large globular set described just above must be a good candidate to start with for a complet solution of this conjecture.

Suppose that the coglobular object $(B^{\bullet}_{C},\delta,\kappa)$ of this article lives in a category $\mathcal{C}$ having small pushouts. According to Proposition 7.2 of the article [2] we can associate to it a coendomorphism operad $Coend(B^{\bullet}_{C})$ over the cartesian monad ω^0 of the strict ∞-categories. This operad is called the *Violet Operad* in the thesis [9] because it is a monochromatic higher operad.

Conjecture 5.5. The operad B^0_C of Batanin, which algebras are his definition of weak ∞-categories, is fractal for the coglobular object

$$\begin{tikzcd}
B^0_C \arrow[r, shift left, "\delta^1_0"] \arrow[r, shift right, "\kappa^1_0"'] \arrow[d, "a^0"] & B^1_C \arrow[r, shift left, "\delta^1_2"] \arrow[r, shift right, "\kappa^2_1"'] \arrow[d, "a^1"] & B^2_C \arrow[r, dotted, shift left] \arrow[r, dotted, shift right] \arrow[d, "a^2"] & B^{n-1}_C \arrow[r, shift left, "\delta^n_{n-1}"] \arrow[r, shift right, "\kappa^n_{n-1}"'] \arrow[d, "a^{n-1}"] & B^n_C \arrow[r, dotted, shift left] \arrow[r, dotted, shift right] \arrow[d, "a^n"] & {} \\
\omega^0(1) \arrow[r, shift left, "\delta^1_0"] \arrow[r, shift right, "\kappa^1_0"'] & \omega^1(1,1) \arrow[r, shift left, "\delta^1_2"] \arrow[r, shift right, "\kappa^2_1"'] & \omega^2(1,1) \arrow[r, dotted, shift left] \arrow[r, dotted, shift right] & \omega^{n-1}(1,1) \arrow[r, shift left, "\delta^n_{n-1}"] \arrow[r, shift right, "\kappa^n_{n-1}"'] & \omega^n(1,1) \arrow[r, dotted, shift left] \arrow[r, dotted, shift right] & {}
\end{tikzcd}$$

described in this article.

A way to prove this conjecture is to replace it with this second conjecture.

Conjecture 5.6. The *violet operad* $Coend(B^{\bullet}_{C})$ is contractible and equipped with a C^0-system.

We believe in Conjecture 5.6 which implies Conjecture 5.5, because examples of fractal operads described in [11] have these properties, that is the *white operad*, the *blue operad*, the *yellow operad*, and the *green operad* of the article [11] have respectively main features of the operad $B^{\bullet}_{G}$ of globular sets, the operad $B^0_{G_u}$ of reflexive globular sets, the operad B^0_M of higher magmas, and the operad $B^0_{M_u}$ of reflexive higher magmas.

Of course if B^0_C is fractal for the coglobular object $(B^{\bullet}_{C},\delta,\kappa)$ of weak higher categories, the standard action associated to it gives the following diagram

$$\begin{tikzcd}
B^0_C \arrow[r, "!_C"] & Coend(B^{\bullet}_{C}) \arrow[r, "Coend((.)\text{-}\mathbb{A}lg)"] & Coend(A^{op}_{C}) \arrow[r, "Coend(Ob(.))"] & End(A_{0,C})
\end{tikzcd}$$

which expresses an action of the operad B^0_C of weak higher categories on the globular object $B^{\bullet}_C$-$\mathbb{A}lg(0)$ in SET of weak higher transformations, and thus gives a weak higher category structure on B^n_C-algebras ($n \in \mathbb{N}$), which means that the globular weak higher category of globular weak higher categories exist.

5.2 The Indigo Operad

In this section we provide a presentation of the ω^n-operads ω^n of strict n-transformations ($n \in \mathbb{N}$) similar to the B^n_C of Section 4 that we denote by $B^n_{S_u}$, and by using our technology of fractality, reformulate the following well known result.

Proposition 5.7. *The strict higher category of the strict higher categories exists.*

Then, surprisingly, this result becomes still a conjecture as in 5.1. Paradoxically the author believes that it is a good clue for the veracity of conjectures in Section 5.1: This reformulation put two problems in the same level of difficulty when kinds of contractibility are involved, and because Proposition 5.7 is true, the ω^0-operad ω^0 must be fractal otherwise Proposition 5.7 is false for the canonical globular object build in the end of Section 2.1.

For each $n > 0$, we are going to define the pointed ω^n-collections with contractible units[16], which are similar to the pointed collections with contractible units defined in Section 3 of the article [11]. We denote by $(\mathbb{R}, \eta, \mu)$ the monad of reflexive globular sets (see Section 3 in [11]). A reflexive ω^n-collection $(C = (C_0, C_1), a = (a_0, a_1), c = (c_0, c_1))$ is an ω^n-collection such that C_0 and C_1 are reflexive globular sets, and a_0 and a_1 are morphisms of reflexive globular sets. The terminal globular set 1 has a natural reflexive globular set structure given by $1^p_m(1(p)) = 1(m)$. Thus, the maps c_0 and c_1 are also morphisms of reflexive globular sets. A pointed ω^n-collection $(C = (C_0, C_1), a = (a_0, a_1), c = (c_0, c_1); (p_0, p_1))$ has contractible units if there exist two monomorphisms $\mathbb{R}(1) \xrightarrow{i_0} C_0$ and $\mathbb{R}(1) \xrightarrow{i_1} C_1$ in $\mathbb{G}lob$

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[16] For $n = 0$, it will be evident to guess the definition of the pointed ω^0-collections with contractible units.

such that we have the following factorizations

$$\begin{tikzcd} & \mathbb{R}(1) \arrow[dr, "i_0"] & \\ 1 \arrow[ur, "\eta(1)"] \arrow[rr, "p_0"'] & & C_0 \end{tikzcd} \qquad \begin{tikzcd} & \mathbb{R}(1) \arrow[dr, "i_1"] & \\ 1 \arrow[ur, "\eta(1)"] \arrow[rr, "p_1"'] & & C_1 \end{tikzcd}$$

such that the induced morphisms

$$\omega^n(1,1) \xleftarrow{a} (\mathbb{R}(1),\mathbb{R}(1)) \xrightarrow{c} (1,1)$$

is a reflexive ω^n-collection. We denote the pointed ω^n-collections with contractible units by $(C,a,c;p,i,(1^p_m)_{0\leqslant p<m})$. A morphism

$$(C,a,c;p,i,(1^p_m)_{0\leqslant p<m}) \xrightarrow{(f,id)} (C',a',c';p',i',(1^p_m)_{0\leqslant p<m})$$

of pointed ω^n-collections with contractible units is given by a morphism of pointed ω^n-collections

$$(C,a,c;p) \xrightarrow{(f,id)} (C',a',c';p')$$

such that $fi=i'$, and $(\mathbb{R}(1),a,c) \xrightarrow{(f,id)} (\mathbb{R}(1),a',c')$ is a morphism of reflexive ω^n-collections. Thus, the morphisms between two ω^n-collections equipped with contractible units preserve this *structure of contractibility on the units.*

Now we are ready to define the category $S_u\omega^n\text{-}\mathbb{C}oll_p$ of pointed strictly contractible ω^n-collections equipped with contractible units: For each integer $n>0$ and each ω^n-collection $(C=(C_0,C_1),a=(a_0,a_1),c=(c_0,c_1))$ we associate another object $\tilde{C}=(\tilde{C_0},\tilde{C_1})$ of the category $\mathbb{G}lob^2$ (see Section 4), which allow us to give the definition of strict contractibility

Definition 5.8. For each integer $n>0$ an ω^n-collection $(C=(C_0,C_1),a=(a_0,a_1),c=(c_0,c_1))$ is strictly contractible if for all integers $k\in\mathbb{N}$, any pair $(x,y)\in\tilde{C_i}(k)$ $(i\in\{0,1\})$ is such that $x=y$. An ω^n-operad is strictly contractible if its underlying ω^n-collection is strictly contractible.

The morphisms between two pointed strictly contractible ω^n-collections equipped with contractible units preserve these *strict contractibility structures* and these *structure of contractibility on the units.* The objects of $S_u\omega^n\text{-}\mathbb{C}oll_p$ are also denoted as $(C, a, c; p, i, (1^p_m)_{0\leqslant p<m})$. It is easy to see that it is a locally presentable category, because it is based on the locally presentable category $\omega^n\text{-}\mathbb{C}oll_p$, and equipped with a strict contractibility structure and with a structure of contractibility on the units, which the operations 1^p_m on the units and their axioms, show easily that $S_u\omega^n\text{-}\mathbb{C}oll_p$ is also projectively sketchable[17]. Also we have the following easy proposition that we can prove by induction on the dimensions of cells.

Proposition 5.9. *If x, y are m-cells $(m \in \mathbb{N})$ of a strictly contractible ω^n-collection $(C = (C_0, C_1), a = (a_0, a_1), c = (c_0, c_1))$ such that $a_i(x) = a_i(y)$ $(i \in \{0, 1\})$ then they are parallel and thus equal.*

We can easily prove that the forgetful functor

$$S_u\omega^n\text{-}\mathbb{C}oll_p \xrightarrow{U'} \omega^n\text{-}\mathbb{C}oll_p$$

is a right adjoint by using basic techniques coming from logic as in [7]. Thus we can apply Proposition 5.5.6 of [3] which shows the monad $\omega^n_{S_u}$ induced by this adjunction has rank. Also U' is monadic by the Beck theorem on monadicity. We write R' for the left adjoint of U':

$$\begin{tikzcd} S_u\omega^n\text{-}\mathbb{C}oll_p \arrow[r, shift left, "U'"] \arrow[r, phantom, "\top"] & \omega^n\text{-}\mathbb{C}oll_p \arrow[l, shift left, "R'"] \end{tikzcd} .$$

Furthermore we can use the result of Kelly as in section 4 to produce with the monadic functor $\omega^n\text{-}\mathbb{O}per \xrightarrow{V} \omega^n\text{-}\mathbb{C}oll_p$ the finitely locally presentable category[18]

$$S_u\omega^n\text{-}\mathbb{O}per := S_u\omega^n\text{-}\mathbb{C}oll_p \underset{\omega^n\text{-}\mathbb{C}oll_p}{\times} \omega^n\text{-}\mathbb{O}per$$

and the monadic functor

$$S_u\omega^n\text{-}\mathbb{O}per \xrightarrow{O} \omega^n\text{-}\mathbb{C}oll_p .$$

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[17] Good references for sketch theory are [1, 3, 4, 15].

[18] $S_u\omega^n\text{-}\mathbb{O}per$ is the category of strictly contractible ω^n-operads with chosen contractible units where in particular morphisms of this category preserve the strict contractibility and the contractibility of the units.

Denote by F the left adjoint to O. If we apply F to the pointed ω^n-collection $(C^n, a^n, c^n; p^n)$ as in 3 we obtain a strictly contractible ω^n-operads $(B^n_{S_u}, a^n, c^n; p^n)$ equipped with contractible units and with a C^n-system. In fact we have the following proposition.[19]

Proposition 5.10. *For each $n \in \mathbb{N}$, we have isomorphisms of operads $B^n_{S_u} \backsimeq \omega^n$.*

This proposition comes from the fact that the morphism

$$B^n_{S_u} \xrightarrow{a^n} \omega^n(1,1)$$

has a section[20], thus is an epimorphism, and also it is a monomorphism, thanks to Proposition 5.9 just above. We can mimic the proof in Section 5.1 to produce another presentation of the coglobular object $(B^\bullet_{S_u}, \delta, \kappa)$ of higher operads for strict higher transformations.

$$\begin{tikzcd}
B^0_{S_u} \arrow[r, shift left, "\delta^1_0"] \arrow[r, shift right, "\kappa^1_0"'] \arrow[d, "a^0"] & B^1_{S_u} \arrow[r, shift left, "\delta^1_2"] \arrow[r, shift right, "\kappa^2_1"'] \arrow[d, "a^1"] & B^2_{S_u} \arrow[r, dotted, shift left] \arrow[r, dotted, shift right] \arrow[d, "a^2"] & B^{n-1}_{S_u} \arrow[r, shift left, "\delta^n_{n-1}"] \arrow[r, shift right, "\kappa^n_{n-1}"'] \arrow[d, "a^{n-1}"] & B^n_{S_u} \arrow[r, dotted, shift left] \arrow[r, dotted, shift right] \arrow[d, "a^n"] & {} \\
\omega^0(1) \arrow[r, shift left, "\delta^1_0"] \arrow[r, shift right, "\kappa^1_0"'] & \omega^1(1,1) \arrow[r, shift left, "\delta^1_2"] \arrow[r, shift right, "\kappa^2_1"'] & \omega^2(1,1) \arrow[r, dotted, shift left] \arrow[r, dotted, shift right] & \omega^{n-1}(1,1) \arrow[r, shift left, "\delta^n_{n-1}"] \arrow[r, shift right, "\kappa^n_{n-1}"'] & \omega^n(1,1) \arrow[r, dotted, shift left] \arrow[r, dotted, shift right] & {}
\end{tikzcd}$$

Proposition 5.10 tell us that the operads of strict higher transformations have similar presentations as those of weak higher transformations, and the coglobular object $(B^\bullet_{S_u}, \delta, \kappa)$ allows to a precise reformulation, with our technology of fractality, of the existence of the strict higher category of strict higher categories. Indeed, as in Section 5.1 we go as follow: Suppose the coglobular object $(B^\bullet_{S_u}, \delta, \kappa)$ is a coglobular object of a category $\mathcal{C}$ having small pushouts. Then, Proposition 7.2 in [2] allows to build an associated coendomorphism higher operad $Coend(B^\bullet_{S_u})$ called the *indigo operad* in [9]. Similar to Section 5.1, we believe the operad $B^0_{S_u}$ of strict higher categories have the fractal property for this coglobular object $(B^\bullet_{S_u}, \delta, \kappa)$. A way to

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[19] $B^n_{S_u}$ is seen here as a cartesian monad over ω^n, and the cartesian identity map of ω^n over itself is a way to see ω^n as the initial object of the category $S_u C^n \omega^n\text{-}\mathbb{O}per$ of strictly contractible ω^n-operads equipped with contractible units and equipped with a C^n-system.

[20] We can build this section by induction on dimensions of cells, thanks to the fact that $B^n_{S_u}$ is strictly contractible and is equipped with a C^n-system.

prove it could be to show that the indigo operad is strictly contractible, is equipped with contractible units, and has a C^0-system. Surprisingly, the author believes that such solution has the same level of difficulty as hypotheses stated in 5.1 for the violet operad. It means that the technology developed in [9–11] to prove fractal phenomenons for higher structures has direct applications when no kinds of contractibilities are involved (like in the article [11]), but leads to non-trivial problems when kinds of contractibility are involved. However we can see that our technology must be a crucial feature for unifying many fractal phenomenons for higher structures, especially if our conjectures are true.

If $B^0_{S_u}$ is fractal for the coglobular object $(B^\bullet_{S_u}, \delta, \kappa)$ of strict higher categories, the standard action associated to it gives the following diagram

$$\begin{tikzcd}
B^0_{S_u} \arrow[r, "!_{S_u}"] & Coend(B^\bullet_{S_u}) \arrow[r, "Coend((.)\text{-}\mathbb{A}lg)"] & Coend(A^{op}_{S_u}) \arrow[r, "Coend(Ob(.))"] & End(A_{0,S_u})
\end{tikzcd}$$

which expresses an action of the operad $B^0_{S_u}$ of strict higher categories on the globular object $B^\bullet_{S_u}$-$\mathbb{A}lg(0)$ in SET of strict (n, ∞)-transformations $(n \in \mathbb{N})$, and thus gives a strict higher category structure on strict n-transformations $(n \in \mathbb{N})$, which means that the strict higher category of strict higher categories exists.

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